p-adic differential equations
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Master theorems for slope factorization

I have used several constructions called "slope factorizations" so far. In this unit, I give a general framework that includes all of them.

Throughout this unit only, rings are unital but *not* necessarily commutative.

1 The master theorems

For R a ring, we define a *norm* on R as in the case of a field, i.e., it is a function $|\cdot| : R \to \mathbb{R}_{\geq 0}$ satisfying the following conditions.

(a) For $r \in R$, $|r| = 0$ if and only if $r = 0$.

(b) For $r, s \in R$, $|r + s| \leq \max\{|r|, |s|\}$.

(c) For $r, s \in R$, $|rs| = |r||s|$.

A *plus-minus decomposition* of R is a direct sum decomposition $R_- \oplus R_0 \oplus R_+$ of the additive group of R such that

$$\begin{aligned} &1 \in R_0, \quad R_0 \cdot R_0 \subseteq R_0, \\ &(R_0 \oplus R_-) \cdot R_- \cdot (R_0 \oplus R_-) \subseteq R_-, \\ &(R_0 \oplus R_+) \cdot R_+ \cdot (R_0 \oplus R_+) \subseteq R_+. \end{aligned}$$

For $r \in R$, write $f_-(r), f_0(r), f_+(r)$ for the components of r under the decomposition.

Theorem 1. *Let R be a ring complete under the norm $|\cdot|$, equipped with a plus-minus decomposition. Then for any $r \in R$ such that*

$$f_0(r) \in R_0^\times, \qquad |f_+(r)| < |f_0(r)|, \qquad |f_-(r)| < |f_0(r)|,$$

there exists a unique factorization $r = s_+ s_-$ with

$$s_+ \in R_0 \oplus R_+, \qquad s_- - 1 \in R_-, \qquad |s_+| \leq |f_0(r)|, \qquad |s_- - 1| < |f_0(r)|.$$

Proof. By multiplying on the left by the inverse of $f_0(r)$, we may reduce to the case where $f_0(r) = 1$, $|f_+(r)| < 1$, $|f_-(r)| < 1$. Put $\eta = \max\{|f_+(r)|, |f_-(r)|\}$.

We first check existence. Put $s_{+,0} = 1 + f_+(r)$, $s_{-,0} = 1 + f_-(r)$. Given $s_{+,l}, s_{-,l}$ such that $s_{+,l} \in R_0 \oplus R_+$, $|s_{+,l} - 1| \leq \eta$, $s_{-,l} - 1 \in R_-$, $|s_{+,l} - 1| \leq \eta$, put $\delta_l = r - s_{+,l} s_{-,l}$. Then set

$$s_{+,l+1} = s_+ + f_0(\delta_l) + f_+(\delta_l), \qquad s_{-,l+1} = s_- + f_-(\delta_l).$$

We then see that

$$\begin{aligned}|\delta_{l+1}| &= |(f_0(\delta_l) + f_+(\delta_l))(1 - s_{-,l}) + (1 - s_{+,l})f_-(\delta_l)| \\ &\leq \eta|\delta_l|.\end{aligned}$$

Thus the sequences $s_{+,l}, s_{-,l}$ converge to limits s_+, s_- with the desired properties.

We next check uniqueness; again, we may assume $f_0(r) = 1$. If $s'_+ s'_-$ is another factorization of the desired form, put $\rho = \max\{|s_+ - s'_+|, |s_- - s'_-|\}$. Put

$$\begin{aligned}x &= (s_+ - s'_+)s_- \\ &= s'_+(s'_- - s_-) \\ &= (s_+ - s'_+) + (s_+ - s'_+)(s_- - 1) \\ &= (s'_- - s_-) + (s'_+ - 1)(s'_- - s_-).\end{aligned}$$

The third expression yields $|f_-(x)| \leq \rho\eta$; the fourth expression yields $|f_0(x)|, |f_+(x)| \leq \rho\eta$. However, the first two expressions together force $|x| = \eta$, which is a contradiction unless $\rho = 0$. □

Theorem 1 works well for rings of polynomials but not for rings of power series; for that, we must add some extra structure. Let R be a ring equipped with a plus-minus decomposition. A *plus-minus grading* on R is a pair of functions

$$\deg_* : R_* \to \mathbb{Z}_{\geq 0} \qquad (* \in \{+, -\})$$

satisfying the following conditions.

(a) For $* \in \{+, -\}$ and $r_1, r_2 \in R_*$, $\deg_*(r_1 + r_2) \leq \max\{\deg_*(r_1), \deg_*(r_2)\}$.

(b) For $* \in \{+, -\}$ and $r_1, r_2 \in R$, $\deg_*(f_*(r_1 r_2)) \leq \deg_*(f_*(r_1)) + \deg_*(f_*(r_2))$.

Theorem 2. *Let R be a ring equipped with a norm $|\cdot|$, a plus-minus decomposition, and a plus-minus grading. Assume that for $n \in \mathbb{Z}_{\geq 0}$ and $* \in \{+, -\}$, the set $\{r_* \in R_* : \deg_*(r_*) \leq n\}$ is complete under $|\cdot|$. Then for any $r \in R$ such that*

$$f_0(r) \in R_0^\times, \qquad |f_+(r)| < |f_0(r)|, \qquad |f_-(r)| < |f_0(r)|,$$

there exists a unique factorization $r = s_+ s_-$ with

$$s_+ \in R_0 \oplus R_+, \qquad s_- - 1 \in R_-, \qquad |s_+| \leq |f_0(r)|, \qquad |s_- - 1| < |f_0(r)|.$$

Proof. The proof is the same as that of Theorem 1, except that we must note that for each l, $\deg_*(s_{*,l}) \leq \deg_*(f_*(r))$. This again allows us to form a limit s_* of $s_{*,l}$, and to proceed as before. □