CHAPTER 18

The p-adic local monodromy theorem

In this chapter, we assert the p-adic local monodromy theorem, and sketch how it can be proved either using deep properties of p-adic differential equations, or using a theory of slope filtrations for Frobenius modules over the Robba ring.

1. Statement of the theorem

REMARK 18.1.1. Recall that we have defined the Robba ring to be

$$\mathcal{R} = \cup_{\alpha \in (0,1)} K\langle \alpha/t, t\}\};$$

that is, $\mathcal{R}$ consists of formal sums $\sum c_i t^i$ which converge in some range $\alpha \leq |t| < 1$, but need not have bounded coefficients. Unlike its subring $\mathcal{E}^\dagger$, $\mathcal{R}$ is not a field; for instance, the element

$$\log(1+t) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} t^i$$

is not invertible (because its Newton polygon has infinitely many slopes). More generally, we have the following easy fact.

LEMMA 18.1.2. *We have* $\mathcal{R}^\times = (\mathcal{E}^\dagger)^\times$.

DEFINITION 18.1.3. Because $\mathcal{R}$ consists of series with possibly unbounded coefficients, it does not carry a natural p-adic topology. The most useful topology on $\mathcal{R}$ is the *LF topology*, which is the direct limit of the Fréchet topology on each $K\langle \alpha/t, t\}\}$ defined by the $|\cdot|_\rho$ for $\rho \in [\alpha, 1)$.

In fact, the ring $\mathcal{R}$ is not even noetherian (this is related to an earlier exercise), but the following useful facts are true; see notes.

PROPOSITION 18.1.4. *For an ideal I of $\mathcal{R}$, the following are equivalent.*

(a) *The ideal I is closed in the LF topology.*
(b) *The ideal I is finitely generated.*
(c) *The ideal I is principal.*

PROPOSITION 18.1.5. *Any finite free module on the half-open annulus with closed inner radius α and open outer radius 1 is represented by a finite free module over $K\langle \alpha/t, t\}\}$, and so corresponds to a finite free module over $\mathcal{R}$. (The first part generalizes to half-open and open annuli with arbitrary boundary radii.)*

DEFINITION 18.1.6. For L a finite separable extension of $\kappa_K((t))$, put

$$\mathcal{R}_L = \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{E}^\dagger_L.$$

We say a finite differential module M over $\mathcal{R}$ is *quasiconstant* if there exists L such that $M \otimes \mathcal{R}_L$ is trivial. We say M is *quasiunipotent* if it is a successive extension of quasiconstant modules; it is equivalent to ask that $M \otimes \mathcal{E}_L^\dagger$ be unipotent (i.e., an extension of trivial differential modules) for some L (exercise).

Quasiunipotent differential modules have many useful properties. For instance, by Proposition 17.2.7, they are all solvable at 1. Another important property is the following.

PROPOSITION 18.1.7. *Let M be a quasiunipotent differential module over $\mathcal{R}$. Then the spaces $H^0(M), H^1(M)$ are finite dimensional, and there is a perfect pairing*

$$H^0(M) \times H^1(M^\vee) \to H^1(M \otimes M^\vee) \to H^1(\mathcal{R}) \cong K\frac{dt}{t}.$$

PROOF. This can be reduced to the unipotent case, for which it is an exercise. $\square$

The following important theorem asserts that many naturally occurring differential modules, including Picard-Fuchs modules, are quasiunipotent. See the notes for further discussion.

THEOREM 18.1.8 (p-adic local monodromy theorem). *Let M be a finite differential module over $\mathcal{R}$ admitting a Frobenius structure for some scalar-preserving Frobenius lift. Then M is quasiunipotent.*

2. An example

It may be worth seeing what Theorem 18.1.8 says in an explicit example. This example was originally considered by Dwork [**Dwo74**]; the analysis given here is due to Tsuzuki [**Tsu98c**, Example 6.2.6], and was cited in the introduction of [**Ked04a**].

EXAMPLE 18.2.1. Let p be an odd prime, put $K = \mathbb{Q}_p(\pi)$ with $\pi^{p-1} = -p$. Let M be the differential module of rank 2 over $\mathcal{R}$ with the action of D on a basis e_1, e_2 given by

$$N = \begin{pmatrix} 0 & t^{-1} \\ \pi^2 t^{-2} & 0 \end{pmatrix}.$$

Then M admits a Frobenius structure; this was shown by explicit calculation in [**Dwo74**], but can also be derived by consideration of a suitable Picard-Fuchs module. Define the tamely ramified extension L of $\kappa_K((t))$ by adjoining u such that $4u^2 = t$, and put

$$u_\pm = 1 + \sum_{n=1}^{\infty} (\pm 1)^n \frac{(2n)!^2}{(32\pi)^n n!^3} u^n \in K\{\{u\}\}.$$

Define the matrix

$$U = \begin{pmatrix} u_+ & u_- \\ u\frac{d}{du}(u_+) + (\frac{1}{2} - \pi u^{-1})u_+ & u\frac{d}{du}(u_-) + (\frac{1}{2} + \pi u^{-1})u_- \end{pmatrix}$$

and use it to change basis; then the action of $\frac{d}{du}$ on the new basis e_+, e_- of $M \otimes \mathcal{R}_L$ is via the matrix

$$\begin{pmatrix} -\frac{1}{2}^y -1 + \pi y^{-2} & 0 \\ 0 & -\frac{1}{2}y^{-1} - \pi y^{-2}. \end{pmatrix}.$$

That is, $M \otimes \mathcal{R}_L$ splits into two differential submodules of rank 1. To render these quasi-constant, we must adjoin to L to L a square root of y (to eliminate the terms $-\frac{1}{2}y^{-1}$) and a root of the polynomial $z^p - z = y^{-1}$ (which by Example 17.2.10 eliminates the terms $\pm\pi y^{-2}$).

By further analysis (carried out in [**Tsu98c**, Example 6.2.6]), one determines that in this example, the generic Newton slopes are 0 and $\log p$, while the special Newton slopes are both $\frac{1}{2}\log p$.

3. The differential approach

There are two general strategies known to be able to prove Theorem 18.1.8. The first is to start with the following result of Christol-Mebkhout, which follows from the p-adic Fuchs theorem for annuli (Theorem 12.6.1).

THEOREM 18.3.1 (Christol-Mebkhout). *Let M be a finite differential module over $\mathcal{R}$ admitting a Frobenius structure for some scalar-preserving Frobenius lift. Suppose that $IR(M \otimes F_\rho) = 1$ for $\rho \in (0,1)$ sufficiently close to 1. Then there exists a positive integer m coprime to p such that $M \otimes \mathcal{R}[t^{1/m}]$ is unipotent.*

Using this and the Christol-Mebkhout decomposition theorem (Theorem 11.5.4), one ultimately reduces to checking Theorem 18.1.8 in rank 1, which one can deduce either from Theorem 17.4.2 or from more elementary considerations.

The reduction process alluded to is not entirely straightforward, and there are multiple ways to organize it. A rather hands-on approach is given by Mebkhout [**Meb02**]; an alternate approach in the language of Tannakian categories is given by André [**And02**]. In either case, a key step is to show that differential modules over $\mathcal{R}$ with Frobenius structures behave like representations of a finite p-group in one important respect: if M is absolutely irreducible (that is, it remains irreducible even after enlarging K), then $\operatorname{rank}(M)$ is a power of p (or at least must be divisible by p if it is greater than 1).

4. The difference approach

The second strategy available for proving Theorem 18.1.8 is to reduce it to Theorem 17.4.2. The main ingredient is the following theorem, analogous to Theorem 13.4.13 but significantly subtler. It was originally proved by Kedlaya in [**Ked04a**] for an absolute Frobenius lift, and in [**Ked07c**] in the form stated here. (See also [**Ked05b**].)

THEOREM 18.4.1 (Slope filtration theorem). *Let M be a finite free difference module over $\mathcal{R}$. Then there exists a unique filtration $0 = M_0 \subset \cdots \subset M_l = M$ by difference submodules with the following properties.*

(a) *Each successive quotient M_i/M_{i-1} is finite free, and is the base extension of a difference module over $\mathcal{E}^\dagger$ which is pure of some norm s_i.*
(b) *We have $s_1 > \cdots > s_l$.*

Note that in order to apply Theorem 18.4.1 to reduce Theorem 18.1.8 to Theorem 17.4.2, one must also check the following lemma.

LEMMA 18.4.2. *Let M be a finite free unit-root difference module over $\mathcal{E}^\dagger$ such that $M \otimes \mathcal{R}$ admits a compatible differential structure. Then this structure is induced by a corresponding differential structure on M itself.*

PROOF. Let N, A be the matrices via which D, ϕ act on a basis of M. Write the commutation relation between N, A in the form $N - pt^{p-1}A\phi(N)A^{-1} = \frac{d}{dt}(A)A^{-1}$. We deduce from Lemma 18.4.3 below that N has entries in $\mathcal{E}^\dagger$. $\square$

LEMMA 18.4.3. *Let A be an $n \times n$ matrix over $\mathfrak{o}_{\mathcal{E}^\dagger}$, and suppose $v \in \mathcal{R}^n$, $w \in (\mathcal{E}^\dagger)^n$ satisfy $v - A\phi(v) = w$. Then $v \in (\mathcal{E}^\dagger)^n$.*

PROOF. Exercise, or see [**Ked07c**, Proposition 1.2.6]. $\square$

5. Applications of the monodromy theorem

The original area of application of the p-adic local monodromy theorem was in the subject of rigid cohomology; the name comes from the fact that it plays a role analogous to the ℓ-adic local monodromy theorem of Grothendieck in the subject of étale cohomology. In particular, Crew [**Cre98**] showed that Theorem 18.1.8 implies the finite dimensionality of the rigid cohomology of a curve with coefficients in an overconvergent F-isocrystal; this was later generalized to arbitrary varieties by Kedlaya [**Ked06**].

Another area of application of the p-adic local monodromy theorem is in p-adic Hodge theory. This will be discussed more thoroughly in Chapter 21.

Theorem 18.1.8 is also needed for the proofs of some more mundane facts about differential modules; it can often be used in lieu of Dwork's trick (Corollary 15.2.4) when working over an annulus instead of a disc. Here is a typical example; see notes for further discussion.

THEOREM 18.5.1. *Let M be a finite free difference module over $R = K\llbracket t \rrbracket_0$ or $\mathcal{E}^\dagger$ admitting a Frobenius structure for an absolute Frobenius lift. Then*

$$H^0(M) = H^0(M \otimes_R \mathcal{E}).$$

PROOF. For the case $R = \mathcal{E}^\dagger$, it is shown in [**Ked04b**] that any F-invariant horizontal section of $M \otimes_R \mathcal{E}$ belongs to M. Here is a quick sketch of the argument. One first uses a technique of de Jong [**dJ98a**] to show that if $v \in H^0(M \otimes_R \mathcal{E})$, then the induced F-equivariant horizontal map $\psi : M^\vee \to \mathcal{E}$ has the property that $\psi^{-1}(\mathcal{E}^\dagger) \neq 0$, and that the generic slopes of $M^\vee/\psi^{-1}(\mathcal{E}^\dagger)$ has all slopes negative. (This argument uses only the Frobenius structure, not the differential structure.) One then uses Theorem 18.1.8 (plus some additional considerations) to show that the short exact sequence

$$0 \to \ker(\psi) \to \psi^{-1}(\mathcal{E}^\dagger) \to \psi^{-1}(\mathcal{E}^\dagger)/\ker(\psi) \to 0$$

must split. This yields $\psi(M) = \mathcal{E}^\dagger$, forcing $v \in H^0(M)$.

To check the claim at hand in the case $R = \mathcal{E}^\dagger$, we may enlarge K to have algebraically closed residue field; then Corollary 13.6.4 implies that $H^0(M \otimes_R \mathcal{E})$ is spanned by one-dimensional fixed subspaces for the Frobenius action. The previous argument shows that any generator of one of these subspaces belongs to M, proving the claim.

For the case $R = K\llbracket t \rrbracket_0$, we may use the previous argument to reduce to checking that $H^0(M) = H^0(M \otimes_R \mathcal{E}^\dagger)$. Since $K\llbracket t \rrbracket_0 = K\{\{t\}\} \cap \mathcal{E}^\dagger$ inside $\mathcal{R}$, this is equivalent to checking that $H^0(M \otimes_R K\{\{t\}\}) = H^0(M \otimes_R \mathcal{R})$. However, this is evident because $M \otimes_R K\{\{t\}\}$ is a trivial differential module by Dwork's trick (Corollary 15.2.4). $\square$

Notes

Proposition 18.1.4 is the essential content of a paper of Lazard [**Laz62**]. Note that it depends on K being spherically complete, and is false otherwise; however, we have assumed in this part that K is discretely valued, so we are safe.

The p-adic local monodromy theorem (Theorem 18.1.8) is often referred to in the literature as "Crew's conjecture", because it emerged from the work of Crew [**Cre98**] on finite dimensionality of rigid cohomology with coefficients in an overconvergent F-isocrystal. The original conjecture only concerned modules such that the differential and Frobenius structures were both defined over $\mathcal{E}^\dagger$; this form was restated in a more geometric form by de Jong [**dJ98b**]. A closer analysis of Crew's conjecture was then given by Tsuzuki [**Tsu98c**], who explained (using Theorem 17.4.2) how Theorem 18.1.8 would follow from a slope filtration theorem [**Tsu98c**, Theorem 5.2.1].

The case of Theorem 18.5.1 with $R = \mathcal{E}^\dagger$ was originally conjectured by Tsuzuki [**Tsu02**, Conjecture 2.3.3]. The case with $R = K\llbracket t \rrbracket_0$ is an older result of de Jong [**dJ98a**]; the arguments in [**Ked04b**] are closely modeled on those of [**dJ98a**], with the key addition being the substitution of Theorem 18.1.8 for Dwork's trick.

In the case of a unit-root Frobenius structure, Theorem 18.5.1 was known prior to the availability of Theorem 18.1.8. It figures in the work of Cherbonnier-Colmez [**CC98**], which we will discuss in Chapter 21 (see Remark 21.2.6); it was also established by Tsuzuki [**Tsu96**, Proposition 4.1.1].

Exercises

(1) Prove Lemma 18.4.3. (Hint: reduce to the case where $|A|_\rho \leq 1$ for $\rho \in [\alpha, 1)$. Then show that $|v|_\rho$ is bounded for $\rho \in [\alpha, 1)$, by comparing $|v|_\rho$ with $|v|_{\rho^{1/q}}$ using Lemma 15.2.1.)