p-adic differential equations
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Newton polygons

In this unit, we then review the theory of Newton polygons for polynomials over fields with valuations. In the process of doing this, we will also generalize to polynomials twisted by a differential operator.

1 Slopes and Newton polygons

A *normed differential field* is a field F equipped with an absolute value $|\cdot|_F$ and a derivation d which is bounded as a linear operator on F over $F_0 = \ker(d)$. If the absolute value is nonarchimedean, corresponding to a valuation v, this means that the quantity

$$r_0 = \min_{f \in F^\times} \{v(d(f)) - v(f)\}$$

is finite. We'll abbreviate "nonarchimedean normed differential field" to "nonarchimedean differential field".

Let F be a nonarchimedean differential field. Let $P(T) = \sum_{i=0}^n P_i T^i$ be a twisted polynomial of degree n over F. Draw the points in $\mathbb{R}^2$ given by

$$\{(-i, v(f_i)) : i = 0, \dots, n, P_i \neq 0\}.$$

Then form the lower convex hull of these points, i.e., take the intersection of every closed halfplane lying above some nonvertical line containing all the points. The boundary of this region is called the *Newton polygon* of P.

Another way to represent the same data is to form the multiset consisting of the slopes of the polygon, each occurring with multiplicity equal to the width of the corresponding segment. The total cardinality is at most n, with equality if and only if $P_0 \neq 0$; in case of a shortfall, we conventionally put in $+\infty$ as a slope with the missing multiplicity. This gives the *slope multiset (multiset of slopes)* of the twisted polynomial.

Keep in mind that we may take $d = 0$, in which case $r_0 = +\infty$; this gives the usual Newton polygon of an untwisted polynomial. A key example with $d \neq 0$ will be the case $F = \mathbb{C}((z))$ with the z-adic valuation and $d = z\frac{d}{dz}$; in this case, $r_0 = 0$.

2 The multiplicativity property

For untwisted polynomials, it was known to Newton (in the case $F = \mathbb{C}((z))$) that for $P, Q \in F[T]$, the slope multiset of PQ is the union of the slope multisets of P and Q. For twisted polynomials, this is only partly true; we must account for the extent the derivation disturbs absolute values, as measured by r_0.

To do this, it is convenient to use yet another representation of the data contained in the Newton polygon. For $r \in \mathbb{R}$, define the *sloped valuation function* v_r on $F\{T\}$ as

$$v_r\left(\sum_i P_i T^i\right) = \min_i \{v(P_i) + ri\}.$$

That is, v_r is the y-intercept of the supporting line of the Newton polygon of slope r. Note that for $P \in F\{T\}$, the function $r \mapsto v_r(P)$ is continuous.

Proposition 1 (Robba). *For $r \leq r_0$ and $P, Q \in F\{T\}$, we have $v_r(PQ) = v_r(P) + v_r(Q)$.*

Proof. By the continuity of $r \mapsto v_r(*)$, it suffices to check the claim for $r < r_0$. Write $P = \sum_i P_i T^i$ and $Q = \sum_j Q_j T^j$; then

$$PQ = \sum_k \left(\sum_{i+j=k} \sum_{h \geq 0} \binom{i+h}{h} P_{i+h} d^h(Q_j) \right) T^k,$$

and hence

$$\begin{aligned} v_r(PQ) &\geq \min_{h,i,j} \{v(P_{i+h}) + v(b_j) + r(i+j) + (v(d^h(Q_j)) - v(Q_j))\} \\ &\geq \min_{h,i,j} \{v(P_{i+h}) + v(Q_j) + r(i+j) + hr_0\} \\ &\geq \min_{h,i,j} \{v(P_{i+h}) + v(Q_j) + r(i+h+j)\}. \end{aligned} \tag{1}$$

This immediately yields $v_r(PQ) \geq v_r(P) + v_r(Q)$. To establish equality, let i_0 and j_0 be the smallest values of i and j which minimize $ri + v(P_i)$ and $rj + v(Q_j)$, respectively; then (1) achieves its minimum for $h = 0, i = i_0, j = j_0$ but not for any other h, i, j with $i + j = i_0 + j_0$. Hence $v_r(PQ) = v_r(P) + v_r(Q)$. □

Corollary 2. *For $r < r_0$, the multiplicity of r as a slope of PQ is the sum of the multiplicities of r as a slope of P and Q.*

Proof. This follows from Proposition 1 and the fact that the left and right endpoints of the segment of slope r in the Newton polygon are the points where the support lines of slightly smaller and slightly larger slope, respectively, touch the polygon. (Note that this means that we cannot deduce the same conclusion for $r = r_0$; see exercises.) □

Corollary 3. *Suppose P factors as $(T - c_1) \cdots (T - c_n)$ where $v(c_i) < r_0$ for $i = 1, \dots, n$. Then the slopes of P are $v(c_1), \dots, v(c_n)$; that is, the Newton polygon computes the valuations of the roots of P (counted with multiplicity).*

Note that one can also prove multiplicativity using the following property.

Proposition 4. *The Newton polygon of any twisted polynomial and its formal adjoint have the same multiplicities for slopes less than r_0.*

Proof. Exercise. (This can also be easily deduced *a posteriori* using Proposition 6 below.) □

In the other direction, one gets good behavior of v_r under the division algorithm provided that you are dividing by a polynomial with all slopes equal to r, and r is not too large.

Lemma 5. *Let $P(T) \in F\{T\}$ be a polynomial whose slopes are all equal to $r < r_0$. Let $S(T) \in F\{T\}$ be any polynomial, and write $S = PQ + R$ with $\deg(R) < \deg(P)$. Then*

$$v_r(S) = \min\{v_r(P) + v_r(Q), v_r(R)\}.$$

Proof. Exercise. □

3 Slope factorizations (Hensel's lemma)

When F is complete for a nonarchimedean absolute value, one has a sort of converse of Proposition 1.

Proposition 6 (Robba)**.** *Let F be a differential field complete for a valuation v. Fix $r < r_0$ and $m \in \mathbb{Z}_{\geq 0}$. Let $R \in F\{T\}$ be a twisted polynomial such that $v_r(R - T^m) > v_r(T^m)$. Then R can be factored uniquely as PQ, where $P \in F\{T\}$ has degree $\deg(R) - m$ and all slopes less than r, $Q \in F\{T\}$ is monic of degree m and has all slopes greater than r, $v_r(P - 1) > 0$, and $v_r(Q - T^m) > v_r(T^m)$.*

Proof. We first check existence. Define sequences $\{P_l\}, \{Q_l\}$ as follows. Define $P_0 = 1$ and $Q_0 = T^m$. Given P_l and Q_l, write

$$R - P_l Q_l = \sum_i a_i T^i,$$

then put

$$X_l = \sum_{i \geq m} a_i T^{i-m}, \qquad Y_l = \sum_{i < m} a_i T^i$$

and set $P_{l+1} = P_l + X_l$, $Q_{l+1} = Q_l + Y_l$. Put $c_l = v_r(R - P_l Q_l) - rm$, so that $c_0 > 0$. Suppose that $v_r(P_l - 1) \geq c_0$, $v_r(Q_l - T^m) \geq c_0 + rm$, and $c_l \geq c_0$. Then visibly $v_r(P_{l+1} - 1) \geq c_0$ and $v_r(Q_{l+1} - T^m) \geq c_0 + rm$; by Proposition 1,

$$\begin{aligned} c_{l+1} &= v_r(R - (P_l + X_l)(Q_l + Y_l)) - rm \\ &= v_r(X_l(T^m - Q_l) + (1 - P_l)Y_l - X_l Y_l) - rm \\ &\geq \min\{c_l + (c_0 + rm), c_0 + (c_l + rm), c_l + (c_l + rm)\} - rm \\ &\geq c_l + c_0. \end{aligned}$$

By induction on l, we deduce that $c_l \geq (l+1)c_0$. Consequently, the sequences $\{P_l\}$ and $\{Q_l\}$ converge under v_r, and their limits P and Q have the desired properties.

We next check uniqueness. Suppose $R = P_1 Q_1$ is a second such factorization; put $c = \min\{v_r(P - P_1), v_r(Q - Q_1) - v_r(T^m)\}$. Put

$$X = R - P_1 Q = (P - P_1)Q = P_1(Q_1 - Q),$$

and suppose $X \neq 0$; then $v_r(X) = c + v_r(T^m)$ by Proposition 1. Write $X = \sum b_k T^k$, and choose k such that $v_r(X) = v_r(b_k T^k)$. The equality

$$X = (P - P_1)T^m + (P - P_1)(Q - T^m)$$

shows that we cannot have $k < m$, while the equality

$$X = Q_1 - Q + (P_1 - 1)(Q_1 - Q)$$

shows that we cannot have $k \geq m$. This contradiction forces $X = 0$, proving $P = P_1, Q = Q_1$ as desired. $\square$

This yields the following.

Theorem 7. *Any monic twisted polynomial $P \in F\{T\}$ admits a unique factorization*

$$P = P_{r_1} \cdots P_{r_m} P_+$$

for some $r_1 < \cdots < r_m < r_0$, where each P_{r_i} is monic with all slopes equal to r_i, and P_+ is monic with all slopes at least r_0.

Proof. We induct on $\deg(P)$. If P has all slopes at least r_0, we may simply set $P_+ = P$; if P has all slopes equal to some $r < r_0$, we may set $P_r = P$. Otherwise, let r_1 be the least slope of P, let r_2 be the next smallest slope, and pick $r \in (r_1, r_2)$. Apply Proposition 6 to P and r, and call the second factor P_{r_1}. Then apply the induction hypothesis to the first factor. $\square$

Theorem 7 also holds with the factors in the reverse order (exercise).

4 Another version of Hensel's lemma

You may be more accustomed to thinking of Hensel's lemma as a statement about lifting factorization of polynomials from $\mathbb{F}_p$ to $\mathbb{Z}_p$. That statement also admits a twisted analogue, also from [Rob80].

Proposition 8. *Assume that $r_0 > 0$. Suppose $R \in \mathfrak{o}_F\{T\}$ and that the reduction of R modulo $\mathfrak{m}_F$ factors into the product $\overline{PQ}$ of two coprime factors. This factorization then lifts to a factorization $R = PQ$ in $\mathfrak{o}_F\{T\}$.*

Proof. Exercise. $\square$

5 Applications in the untwisted case

We will apply the above results to differential modules in the next unit, but for the rest of this unit, take $d = 0$. We use the aforementioned properties of Newton polygons (which in the untwisted case should be familiar) to tie up the loose ends left in our earlier discussion of extensions of nonarchimedean absolute values. Note that $r_0 = \infty$ in the untwisted case, so the results do not carry any restrictions on slopes.

We first check that if F is a complete nonarchimedean field, then any finite extension E of F admits an extension of $|\cdot|$ to an absolute value on E. If E' is a field intermediate between F and E, we may first extend the absolute value to E' and then to E. Consequently, it suffices to check the case where $E = F(\alpha)$ for some $\alpha \in E$, that is, $E \cong F[T]/(P(T))$ for some monic irreducible polynomial $P \in F[T]$ (the minimal polynomial of α). Apply Theorem 7; since $P(T)$ cannot factor nontrivially, we deduce that P must have a single slope r. We now define an absolute value on E as follows: for $\beta = c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1}$, with $n = \deg(P) = [E : F]$, put

$$|\beta|_E = \max_i \{|c_i| e^{-ri}\}.$$

That is, take $|\beta|_E$ to be the e^{-r}-Gauss norm of the polynomial $c_0 + c_1 T + \cdots + c_{n-1}T^{n-1}$. The multiplicativity of $|\cdot|_E$ is then a consequence of Lemma 5.

We next check that the completion E of an algebraically closed nonarchimedean field F is itself algebraically closed. Let $P(T) \in E[T]$ be a monic polynomial of degree d.

Lemma 9. *With notation as above, for any $\epsilon > 0$, we can find $z \in F$ such that $|z| \leq |P(0)|^{1/d}$ and $|P(z)| < \epsilon$.*

Proof. If $P(0) = 0$ we may pick $z = 0$, so assume $P(0) \neq 0$. Put $P = T^d + \sum_{i=0}^{d-1} P_i T^i$. For any $\delta > 0$, we can pick a polynomial $Q = T^d + \sum_{i=0}^{d-1} Q_i d^i \in F[T]$ with $|Q_i - P_i| < \delta$ for $i = 0, \dots, d-1$.

Now assume $\delta < \min\{|P_0|, \epsilon, \epsilon/|P_0|\}$, so that $|Q_0| = |P_0|$. By Corollary 3, we can find a root $z \in F$ of Q_0 with $|z| \leq |Q_0|^{1/d} = |P_0|^{1/d}$. We now have

$$|P(z)| = |(P - Q)(z)| \leq \delta \max\{1, |z|\}^d \leq \delta \max\{1, |P(0)|\} < \epsilon,$$

as desired. □

Define a sequence of polynomials $P_0, P_1, \dots$ as follows. Put $P_0 = P$. Given P_i, apply Lemma 9 to construct z_i with $|z_i| \leq |P_i(0)|^{1/d}$ and $|P_i(z_i)| < 2^{-i}$, then set $P_{i+1}(T) = P_i(T + z_i)$ so that $P_{i+1}(0) = P_i(z_i)$. If some P_i satisfies $P_i(0) = 0$, then $z_0 + \cdots + z_{i-1}$ is a root of P. Otherwise, we get an infinite sequence $z_0, z_1, \dots$ such that $z_0 + z_1 + \cdots$ converges to a root of P.

6 Notes

Newton polygons for differential operators were considered by Dwork and Robba [DR77, §6.2.3]; the first systematic treatment seems to have been made by Robba [Rob80], and our results are taken from there. The proofs of Proposition 1 and 6 were self-plagiarized from [Ked07, Lemma 3.1.5 and Proposition 3.2.2].

7 Exercises

1. Prove Proposition 4.

2. Prove Lemma 5.

3. Give an example with $F = \mathbb{C}((z))$ and $d = z\frac{d}{dz}$ to show that the conclusion of Corollary 2 need not hold for $r = r_0$, even though Proposition 1 holds for $r = r_0$.

4. Deduce the analogue of Theorem 7 with the factors in the opposite order, *without* doing any recalculation. (Hint: the key word here is "opposite".)

5. State and prove a precise version of the assertion that "the roots of a polynomial over a complete algebraically closed nonarchimedean field vary continuously in the coefficients."

6. Prove Proposition 8.