CHAPTER 21

p-adic Hodge theory

In this chapter, we describe an analogue of the construction of Chapter 17 for p-adic representations of the absolute Galois group of a mixed characteristic local field. Beware that our presentation is historically inaccurate; see the notes.

HYPOTHESIS 21.0.1. Throughout this chapter, let K be a finite extension of $\mathbb{Q}_p$, let V be a finite dimensional $\mathbb{Q}_p$-vector space, and let $\tau : G_K \to \mathrm{GL}(V)$ be a continuous homomorphism for the p-adic topology on V.

1. A few rings

DEFINITION 21.1.1. Put $K_n = K(\zeta_{p^n})$ and $K_\infty = \cup_n K_n$. Let $F = \mathrm{Frac}\, W(\kappa_K)$ and F' be the maximal subfields of K and K_∞, respectively, which are unramified over $\mathbb{Q}_p$. Put $H_K = G_{K_\infty}$ and $\Gamma_K = G_{K_\infty/K} = G_K/H_K$.

DEFINITION 21.1.2. Put $\mathfrak{o} = \mathfrak{o}_{\mathbb{C}_p}$. Let $\tilde{\mathbf{E}}^+$ be the inverse limit of the system

$$\cdots \to \mathfrak{o}/p\mathfrak{o} \to \mathfrak{o}/p\mathfrak{o}$$

in which each map is the p-power Frobenius (which is a ring homomorphism). More explicitly, the elements of $\tilde{\mathbf{E}}^+$ are sequences $(x_0, x_1, \dots)$ of elements of $\mathfrak{o}/p\mathfrak{o}$ for which $x_{n+1}^p = x_n$ for all n. In particular, for any nonzero $x \in \tilde{\mathbf{E}}^+$, the quantity $p^n v_p(x_n)$ is the same for all n for which $x_n \neq 0$; we call this quantity $v(x)$, and put conventionally $v(0) = +\infty$. Choose $\epsilon = (\epsilon_0, \epsilon_1, \dots) \in \tilde{\mathbf{E}}^+$ with $\epsilon_0 = 1$ and $\epsilon_1 \neq 1$.

The following observations were made by Fontaine and Wintenberger [**FW79**].

PROPOSITION 21.1.3. *The following are true.*

(a) *The ring $\tilde{\mathbf{E}}^+$ is a domain in which $p = 0$, with fraction field $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}^+[\epsilon^{-1}]$.*
(b) *The function $v : \tilde{\mathbf{E}}^+ \to [0, +\infty]$ extends to a valuation on $\tilde{\mathbf{E}}$, under which $\tilde{\mathbf{E}}$ is complete and $\mathfrak{o}_{\tilde{\mathbf{E}}} = \tilde{\mathbf{E}}^+$.*
(c) *The field $\tilde{\mathbf{E}}$ is the algebraic closure of $\kappa_K((\epsilon - 1))$. (The embedding of $\kappa_K((\epsilon - 1))$ into $\tilde{\mathbf{E}}$ exists because $v(\epsilon - 1) = p/(p - 1) > 0$.)*

DEFINITION 21.1.4. Let $\tilde{\mathbf{A}}$ be the ring of Witt vectors of $\tilde{\mathbf{E}}$, i.e., the unique complete discrete valuation ring with maximal ideal p and residue field $\tilde{\mathbf{E}}$. The uniqueness follows from the fact that $\tilde{\mathbf{E}}$ is algebraically closed, hence perfect. In particular, the p-power Frobenius on $\tilde{\mathbf{E}}$ lifts to an automorphism ϕ of $\tilde{\mathbf{A}}$.

DEFINITION 21.1.5. Each element of $\tilde{\mathbf{A}}$ can be uniquely written as a sum $\sum_{n=0}^{\infty} p^n [x_n]$, where $x_n \in \tilde{\mathbf{E}}$ and $[x_n]$ denotes the *Teichmüller lift* of x_n (the unique lift of x_n that has a p^m-th root in $\tilde{\mathbf{A}}$ for all positive integers m); note that $\phi([x]) = [x^p] = [x]^p$. We may thus equip $\tilde{\mathbf{A}}$ with a *weak topology*, in which a sequence $x_m = \sum_{n=0}^{\infty} p^n [x_{m,n}]$ converges to zero if for each n, $v(x_{m,n}) \to \infty$ as $m \to \infty$. Let $\mathbf{A}_{\mathbb{Q}_p}$ be the completion of $\mathbb{Z}_p[([\epsilon]-1)^{\pm}]$ in $\tilde{\mathbf{A}}$ for the weak topology; as a topological ring, it is isomorphic to the ring $\mathfrak{o}_{\mathcal{E}}$ defined over the base field $\mathbb{Q}_p$ with its own weak topology. It is also ϕ-stable because $\phi([\epsilon]) = [\epsilon]^p$.

Definition 21.1.6. Let $\mathbf{A}$ be the completion of the maximal unramified extension of $\mathbf{A}_{\mathbb{Q}_p}$, viewed as a subring of $\tilde{\mathbf{A}}$. Put

$$\mathbf{A}_K = (\mathbf{A} \cap \tilde{\mathbf{B}})^{H_K},$$

where the right side makes sense because we have made all the rings so far in a functorial fashion, so that they indeed carry a G_K-action. Note that $\mathbf{A}_K$ can be written as a ring of the form $\mathfrak{o}_{\mathcal{E}}$, but with coefficients in K' rather than in $\mathbb{Q}_p$.

Definition 21.1.7. For any ring denoted with a boldface A so far, define the corresponding ring with $\mathbf{A}$ replaced by $\mathbf{B}$ by tensoring over $\mathbb{Z}_p$ with $\mathbb{Q}_p$. For instance, $\tilde{\mathbf{B}} = \tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the fraction field of $\tilde{\mathbf{A}}$.

2. (ϕ, Γ)-modules

We are now ready to describe the mechanism, introduced by Fontaine, for converting Galois representations into modules over various rings equipped with much simpler group actions.

Definition 21.2.1. Recall that V is a finite-dimensional vector space equipped with a continuous G_K-action. Put

$$D(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B})^{H_K};$$

by Hilbert's Theorem 90, $D(V)$ is a finite dimensional $\mathbf{B}_K$-vector space, and the natural map $D(V) \otimes_{\mathbf{B}_K} \mathbf{B} \to V \otimes_{\mathbb{Q}_p} \mathbf{B}$ is an isomorphism. Since we only took H_K-invariants, $D(V)$ retains a semilinear action of $G_K/H_K = \Gamma_K$; it also inherits an action of ϕ from $\mathbf{B}$. That is, $D(V)$ is a (ϕ, Γ)-*module* over $\mathbf{B}_K$, i.e., a finite free $\mathbf{B}_K$-module equipped with semilinear ϕ and Γ_K-actions which commute with each other. It is also *étale*, which is to say the ϕ-action is étale (unit-root); as in Definition 17.2.5, this is because one can find a G_K-invariant lattice in V.

Theorem 21.2.2 (Fontaine). *The functor D, from the category of continuous representations of G_K on finite dimensional $\mathbb{Q}_p$-vector spaces to the category of étale (ϕ, Γ)-modules over $\mathbf{B}_K$, is an equivalence of categories.*

Proof. From $D(V)$, one can recover

$$V = (D(V) \otimes_{\mathbf{B}_K} \mathbf{B})^{\phi=1}.$$

□

Theorem 21.2.2 was refined by Cherbonnier and Colmez as follows [**CC98**].

Definition 21.2.3. Let $\mathbf{B}^{\dagger}_{\mathbb{Q}_p}$ be the image of $\mathcal{E}^{\dagger}$ under the identification of $\mathcal{E}$ (with coefficients in $\mathbb{Q}_p$) with $\mathbf{B}_{\mathbb{Q}_p}$ sending t to $[\epsilon]-1$. Let $\mathbf{B}^{\dagger}_K$ be the integral closure of $\mathbf{B}^{\dagger}_{\mathbb{Q}_p}$ in $\mathbf{B}_K$. Again, $\mathbf{B}^{\dagger}_K$ carries actions of ϕ and Γ_K.

Definition 21.2.4. Let $\mathbf{A}^\dagger$ be the set of $x = \sum_{n=0}^\infty p^n [x_n] \in \tilde{\mathbf{A}}$ such that $\liminf_{n \to \infty} \{v(x_n)/n\} > -\infty$. Define

$$D^\dagger(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}^\dagger)^{H_K};$$

it is an étale (ϕ, Γ)-module over $\mathbf{B}_K^\dagger$.

The following is the main result of [**CC98**].

Theorem 21.2.5 (Cherbonnier-Colmez). *The functor $D^\dagger$, from the category of continuous representations of G_K on finite dimensional $\mathbb{Q}_p$-vector spaces to the category of étale (ϕ, Γ)-modules over $\mathbf{B}_K^\dagger$, is an equivalence of categories.*

Remark 21.2.6. By Theorem 21.2.2, it suffices to check that the base extension functor from étale (ϕ, Γ)-modules over $\mathbf{B}_K^\dagger$ to étale (ϕ, Γ)-modules over $\mathbf{B}_K$ is an equivalence. The full faithfulness of this functor is elementary; it follows from Lemma 18.4.6. The essential surjectivity is much deeper; it amounts to the fact that the natural map

$$D^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}^\dagger \to V \otimes_{\mathbb{Q}_p} \mathbf{B}^\dagger$$

is an isomorphism. Verifying this requires developing an appropriate analogy to Sen's theory of decompletion; this analogy has been developed into a full abstract Sen theory by Berger and Colmez [**BC07**].

A further variant was proposed by Berger [**Brg02**].

Definition 21.2.7. Using the identification $\mathbf{B}_{\mathbb{Q}_p}^\dagger \cong \mathcal{E}^\dagger$, put

$$\mathbf{B}_{\mathrm{rig},K}^\dagger = \mathbf{B}_K^\dagger \otimes_{\mathbf{B}_{\mathbb{Q}_p}^\dagger} \mathcal{R}.$$

Note that $\mathbf{B}_{\mathrm{rig},K}^\dagger$ admits continuous extensions (for the LF-topology) of the actions of ϕ and Γ_K. Define

$$D_{\mathrm{rig}}^\dagger(V) = D^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_{\mathrm{rig},K}^\dagger;$$

it is an étale (ϕ, Γ)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$.

Theorem 21.2.8 (Berger). *The functor $D_{\mathrm{rig}}^\dagger$, from the category of continuous representations of G_K on finite dimensional $\mathbb{Q}_p$-vector spaces to the category of étale (ϕ, Γ)-modules over $\mathbf{B}_{\mathrm{rig},K}^\dagger$, is an equivalence of categories.*

Remark 21.2.9. The principal content in Theorem 21.2.8 is that the base extension functor from étale ϕ-modules over $\mathcal{E}^\dagger$ to étale ϕ-modules over $\mathcal{R}$ is fully faithful; this is elementary (see exercises). The essential surjectivity of the functor is almost trivial, since étaleness of the ϕ-action is defined over the Robba ring by base extension from $\mathcal{E}^\dagger$. One needs only check that the Γ_K-action also descends to any étale lattice, but this is easy (it is similar to Lemma 18.4.4).

3. Galois cohomology

Since the functor D and its variants lose no information about Galois representations, it is unsurprising that they can be used to recover basic invariants of a representation, such as Galois cohomology.

Definition 21.3.1. Assume for simplicity that Γ_K is procyclic; this only eliminates the case where $p = 2$ and $\{\pm 1\} \subset \Gamma$, for which see Remark 21.3.2 below. Let γ be a topological generator of Γ. Define the *Herr complex* over $\mathbf{B}_K$ associated to V as the complex (with the first nonzero term placed in degree zero)

$$0 \to D(V) \to D(V) \oplus D(V) \to D(V) \to 0$$

with the first map being $m \mapsto ((\phi - 1)m, (\gamma - 1)m)$ and the second map being $(m_1, m_2) \to (\gamma - 1)m_1 - (\phi - 1)m_2$. (The fact that this is a complex follows from the commutativity between ϕ and γ.) Similarly, define the Herr complex over $\mathbf{B}_K^\dagger$ or $\mathbf{B}_{\mathrm{rig},K}^\dagger$ by replacing $D(V)$ by $D^\dagger(V)$ or $D_{\mathrm{rig}}^\dagger(V)$, respectively.

Remark 21.3.2. A more conceptual description, which also covers the case where Γ_K need not be profinite, is that one takes the total complex associated to

$$0 \to C^\cdot(\Gamma_K, D(V)) \stackrel{\phi-1}{\to} C^\cdot(\Gamma_K, D(V)) \to 0.$$

One might think of this as the "monoid cohomology" of $\Gamma_K \times \phi^{\mathbb{Z}_{\geq 0}}$ acting on $D(V)$.

Theorem 21.3.3. *The cohomology of the Herr complex computes the Galois cohomology of V.*

Proof. For $\mathbf{B}_K$, the desired result was established by Herr [**Her98**]. The argument proceeds in two steps. One first takes cohomology of the Artin-Schreier sequence

$$0 \to \mathbb{Q}_p \to \mathbf{B} \stackrel{\phi-1}{\to} \mathbf{B} \to 0$$

after tensoring with V. This reduces the claim to the fact that the inflation homomorphisms

$$H^i(\Gamma_K, D(V)) \to H^i(G_K, V \otimes_{\mathbb{Q}_p} \mathbf{B})$$

are bijections; this is proved by adapting a technique introduced by Sen.

For $\mathbf{B}_K^\dagger$ and $\mathbf{B}_{\mathrm{rig},K}^\dagger$, the desired result was established by Liu [**Liu07**]; this proceeds by comparison with the original Herr complex rather than by imitating the above argument, though one could probably do that also. $\square$

Remark 21.3.4. As is done in [**Her98, Liu07**], one can make Theorem 21.3.3 more precise. For instance, the construction of Galois cohomology is functorial; there is also an interpretation in the Herr complex of the cup product in cohomology.

Remark 21.3.5. One can also use the Herr complex to recover Tate's fundamental theorems about Galois cohomology (finite dimensionality, Euler-Poincaré characteristic formula, local duality). This was done by Herr in [**Her01**].

4. Differential equations from (ϕ, Γ)-modules

One of the original goals of p-adic Hodge theory was to associate finer invariants to p-adic Galois representations, so as for instance to distinguish those representations which arose in geometry (i.e., from the étale cohomology of varieties over K). This was originally done using a collection of "period rings" introduced by Fontaine; more recently, Berger's work has demonstrated that one can reproduce these constructions using (ϕ, Γ)-modules. Here is a brief description of an example that shows the relevance of p-adic differential equations to this study. We will make reference to Fontaine's rings $\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{st}}$ without definition, for which see [**Brg04**].

Definition 21.4.1. Let $\chi : \Gamma_K \to \mathbb{Z}_p^\times$ denote the cyclotomic character; that is, for all nonnegative integers m and all $\gamma \in \Gamma_K$,

$$\gamma(\zeta_{p^m}) = \zeta_{p^m}^{\chi(\gamma)}.$$

For $\gamma \in \Gamma_K$ sufficiently close to 1, we may compute

$$\nabla = \frac{\log(\gamma)}{\log \chi(\gamma)}$$

as an endomorphism of $D(V)$, using the power series for $\log(1+x)$. The result does not depend on γ.

Remark 21.4.2. If one views Γ_K as a one-dimensional p-adic Lie group over $\mathbb{Z}_p$, then ∇ is the action of the corresponding Lie algebra.

Definition 21.4.3. Note that ∇ acts on $\mathbf{B}^\dagger_{\mathrm{rig},K}$ with respect to

$$f \mapsto [\epsilon] \log[\epsilon] \frac{df}{d[\epsilon]}.$$

As a result, it does not induce a differential module structure with respect to $\frac{d}{dt}$ on $D(V)$, but only on $D(V) \otimes \mathbf{B}^\dagger_{\mathrm{rig},K}[(\log[\epsilon])^{-1}]$. We say that V is *de Rham* if there exists a differential module with Frobenius structure M over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and an isomorphism

$$D(V) \otimes \mathbf{B}^\dagger_{\mathrm{rig},K}[(\log[\epsilon])^{-1}] \to M \otimes \mathbf{B}^\dagger_{\mathrm{rig},K}[(\log[\epsilon])^{-1}]$$

of differential modules with Frobenius structure.

One then has the following results of Berger [**Brg02**].

Theorem 21.4.4 (Berger). (a) *The representation V is de Rham if and only if it is de Rham in Fontaine's sense, i.e., if*

$$D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}$$

satisfies

$$D_{\mathrm{dR}}(V) \otimes_K \mathbf{B}_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}.$$

(b) *Suppose that V is de Rham. Then V is semistable in Fontaine's sense, i.e.,*

$$D_{\mathrm{st}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_K}$$

satisfies

$$D_{\mathrm{st}}(V) \otimes_F \mathbf{B}_{\mathrm{st}} \cong V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}},$$

if and only if there exists M as in Definition 21.4.3 which is unipotent.

Applying Theorem 18.1.8 then yields the following corollary, which was previously a conjecture of Fontaine [**Fon94**, 6.2].

Corollary 21.4.5 (Berger). *Every de Rham representation is potentially semistable, i.e., becomes semistable upon restriction to G_L for some finite extension L of K.*

Remark 21.4.6. The term "de Rham" is meant to convey the fact that if $V = H^i_{\mathrm{et}}(X \times_K K^{\mathrm{alg}}, \mathbb{Q}_p)$ for X a smooth proper variety over K, then V is de Rham and you can use the aforementioned constructions to recover $H^i_{\mathrm{dR}}(X, K)$ functorially from V (solving Grothendieck's "problem of the mysterious functor"). See [**Brg04**] for more of the story.

5. Beyond Galois representations

The category of arbitrary (ϕ, Γ)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ turns out to have its own representation-theoretic interpretation; it is equivalent to the category of *B-pairs* introduced by Berger [**Brg07a**]. One can associate "Galois cohomology" to such objects using the Herr complex; it has been shown by Liu [**Liu07**] that the analogues of Tate's theorems (see Remark 21.3.5) still hold. These functors can be interpreted as the derived functors of $\mathrm{Hom}(D^\dagger_{\mathrm{rig}}(V_0), \cdot)$ for V_0 the trivial representation [**Ked07f**, Appendix].

One may wonder why one should be interested in (ϕ, Γ)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ if ultimately one has in mind an application concerning only Galois representations. One answer is that converting Galois representations into (ϕ, Γ)-modules exposes extra structure that is not visible without the conversion.

Definition 21.5.1 (Colmez). We say V is *trianguline* if $D^\dagger_{\mathrm{rig}}(V)$ is a successive extension of (ϕ, Γ)-modules of rank 1 over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. The point is that these need not be étale, so V need not be a successive extension of representations of dimension 1.

The trianguline representations have the dual benefits of being relatively easy to classify, and somewhat commonplace. On one hand, Colmez [**Col07**] classified the two-dimensional trianguline representations of $G_{\mathbb{Q}_p}$; the classification includes a parameter (the $\mathcal{L}$-invariant) relevant to p-adic L-functions. On the other hand, a result of Kisin [**Kis03**] shows that the Galois representations associated to many classical modular forms are trianguline.

Notes

Our presentation here is largely a summary of Berger's [**Brg04**], which we highly recommend.

A variant of the theory of (ϕ, Γ)-modules was introduced by Kisin [**Kis06**], using the Kummer tower $K(p^{1/p^n})$ instead of the cyclotomic tower $K(\zeta_{p^n})$. This leads to certain advantages, particularly when studying crystalline representations. Kisin's work is based on an earlier paper of Berger [**Brg07b**]; both of these use slope filtrations (as in Theorem 18.4.1) to recover a theorem of Colmez-Fontaine classifying semistable Galois representations in terms of certain linear algebraic data.

After [**Brg02**] appeared, Fontaine succeeded in giving a direct proof of Corollary 21.4.5 (i.e., not going through p-adic differential equations). We do not have a reference for this.

Exercises

(1) (Compare [**Tsu96**, Proposition 2.2.2].) Let A be an $n \times n$ matrix over $\mathfrak{o}_{\mathcal{E}^\dagger}$, and suppose $v \in \mathcal{E}^n$, $w \in (\mathcal{E}^\dagger)^n$ satisfy $Av - \phi(v) = w$. Then $v \in (\mathcal{E}^\dagger)^n$. This gives a direct proof of some cases of Theorem 18.5.1, in the spirit of Lemma 18.4.6. (Hint: reduce to the case where $|A|_\rho \leq 1$ for some $\rho \in (0,1)$ for which $|w|_\rho < \infty$. Then use $|w|_\rho$ to bound the terms of $v = \sum_i v_i t^i$ for which $|v_i| \geq c$.)