p-adic differential equations
18.787, Kiran S. Kedlaya, MIT, fall 2007
Quasiunipotent differential modules

In this unit, we construct a class of examples of differential modules on open annuli which are solvable at a boundary. In the process, we illustrate a numerical relationship between wild ramification in positive characteristic and convergence of solutions of p-adic differential equations. We also state the p-adic local monodromy theorem, for differential modules with Frobenius structure on an annulus, and prove the rank 1 case.

Throughout this unit, we assume that our complete nonarchimedean field carries a discrete valuation (e.g., finite extensions of $\mathbb{Q}_p$ are okay but not $\mathbb{C}_p$). Getting rid of this assumption throws in a number of subtleties which we will not address here.

Notation: for E/F a Galois extension of fields, write $G_{E/F}$ for $\mathrm{Gal}(E/F)$. If $E = F^{\mathrm{sep}}$, write G_F instead, to mean the absolute Galois group.

Also, note that I haven't added all references and details; I plan to put a bit more when I fold this into the compiled notes. (That will be true for the remainder of the course.)

1 Some key rings

Recall that we defined the ring $\mathcal{E}$ as the completion of $\mathfrak{o}_K((t)) \otimes_{\mathfrak{o}_K} K$ for the 1-Gauss norm

$$\left| \sum_{i \in \mathbb{Z}} c_i t^i \right|_1 = \sup_i \{|c_i|\}.$$

Besides the p-adic topology, it is natural to consider also the *weak topology* on $\mathcal{E}$, in which a sequence converges to 0 if it does so in the t-adic topology on $\mathcal{E}/\mathfrak{m}_K^m \mathfrak{o}_{\mathcal{E}}$ for each $m \in \mathbb{Z}$. Note that $\mathcal{E}$ is complete for both topologies.

Because K carries a discrete valuation, the supremum defining the Gauss norm of a nonzero element $x = \sum x_i t^i \in \mathcal{E}$ is achieved by some i. If j is the least such index, then the sum

$$x_j^{-1} t^{-j} \sum_{l=0}^{\infty} (1 - x_j^{-1} t^{-j} x)^l$$

converges in the weak topology (but not in the p-adic topology!) to an inverse of x. That is, $\mathcal{E}$ is a discrete complete nonarchimedean field with residue field $\kappa_K((t))$.

Put

$$\mathcal{E}^\dagger = \bigcup_{\alpha \in (0,1)} K\langle \alpha/t, t]\!]_0;$$

that is, $\mathcal{E}^\dagger$ consists of formal sums $\sum c_i t^i$ which have bounded coefficients and converge in some range $\alpha \leq |t| < 1$.

Lemma 1. *(a) The ring $\mathcal{E}^\dagger$ is a field.*

(b) Under the norm $|\cdot|_1$, the valuation ring $\mathfrak{o}_{\mathcal{E}^\dagger}$ is a local ring with maximal ideal $\mathfrak{m}_K \mathfrak{o}_{\mathcal{E}^\dagger}$.

(c) The pair $(\mathfrak{o}_{\mathcal{E}^\dagger}, \mathfrak{m}_K \mathfrak{o}_{\mathcal{E}^\dagger})$ is henselian.

This last property implies that finite separable extensions of $\kappa_{\mathcal{E}^\dagger} = \kappa_K((t))$ lift functorially to finite étale extensions of $\mathfrak{o}_{\mathcal{E}^\dagger}$ (and to unramified extensions of $\mathcal{E}^\dagger$). In particular, the maximal unramified extension $\mathcal{E}^{\dagger,\mathrm{unr}}$ carries an action of $G_{\kappa_K((t))}$.

Proof. The proof of (a) uses the same construction as for $\mathcal{E}$, except that the series converges under $|\cdot|_\alpha$ for some $\alpha < 1$. From this, (b) is straightforward. The proof of (c) is to reduce to working in some $K\langle \alpha/t, t]\!]_0$ and use the fact that the latter ring is complete for the Fréchet topology generated by $|\cdot|_\alpha$ and $|\cdot|_1$. □

2 Finite representations and differential modules

Let V be a finite dimensional vector space over K, and let $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ be a continuous homomorphism for the *discrete* topology on $\mathrm{GL}(V)$. That is, τ factors through $G_{L/\kappa_K((t))}$ for some finite separable extension L of $\kappa_K((t))$.

Let $\mathcal{E}^\dagger_L$ be the finite unramified extension of $\mathcal{E}^\dagger$ corresponding to L; then $G_{\kappa_K((t))}$ acts on $\mathcal{E}^\dagger_L$ with fixed field $\mathcal{E}^\dagger$. (Minor weirdness: by the Cohen structure theorem, L can always be written as a power series field $\lambda((u))$, and similarly for $\mathcal{E}^\dagger_L$. But if L induces an inseparable residue field extension, then you can't ensure that κ_K can be contained in λ. I recommend not worrying about this unless you really have to.)

Let us view $V \otimes_K \mathcal{E}^\dagger_L$ as a $G_{\kappa_K((t))}$-module with the action on the first factor coming from τ and the action on the second factor as above. Put

$$D^\dagger(V) = (V \otimes_K \mathcal{E}^\dagger_L)^{G_{\kappa_K((t))}}.$$

Lemma 2. *The space $D^\dagger(V)$ is an $\mathcal{E}^\dagger$-vector space of dimension $\dim_K(V)$.*

Proof. This is a consequence of the nonabelian version of Hilbert's Theorem 90: for any finite Galois extension E/F of fields, the nonabelian cohomology set $H^1(G_{E/F}, \mathrm{GL}_n(E))$ is trivial. □

Note that $\frac{d}{dt}$ extends uniquely to $\mathcal{E}^\dagger_L$, and hence to $D^\dagger(V)$ by taking the action on V to be trivial. Since the action of $\frac{d}{dt}$ commutes with the Galois action, we also obtain an action on $D^\dagger(V)$. That is, $D^\dagger(V)$ is a differential module over $\mathcal{E}^\dagger$.

Note that there is a sense in which it makes sense to compute the subsidiary radii of $D^\dagger(V) \otimes F_\rho$ for $\rho \in (0,1)$ sufficiently close to 1. Namely, realize $D^\dagger(V)$ as a differential module over $K\langle \alpha/t, t]\!]_0$ for some α and compute there. Beware that any two such realizations for a given α need only become isomorphic over $K\langle \beta/t, t]\!]_0$ for some $\beta \in [\alpha, 1)$. However, the following statement is unambiguous.

Proposition 3. *The generic radius of convergence of $D^\dagger(V) \otimes \mathcal{E}$ is equal to 1. Consequently (by continuity of generic radius of convergence), $D^\dagger(V)$ is solvable at 1.*

Proof. This can be shown directly, but it also follows from the existence of a Frobenius structure on $D^\dagger(V)$. Namely, fix any Frobenius lift ϕ on $\mathcal{E}^\dagger$; then ϕ extends uniquely to $\mathcal{E}^\dagger_L$. Let ϕ act on $V \otimes_K \mathcal{E}^\dagger_L$ using the trivial action on the first factor; this action commutes with the Galois action, so we get a ϕ-action on $D^\dagger(V)$ compatible with the derivation. □

Note that the Frobenius structure constructed in the previous proof is pure of slope 1 (i.e., is *unit-root*), because one can pick a Galois-stable lattice in V and do everything integrally. This will allow us to form a converse assertion; see below.

3 Ramification and differential slopes

There is a close relationship between $R(D^\dagger(V) \otimes F_\rho)$ and wild ramification of the representation V. To explain this, I need to recall a bit of classical ramification theory for local fields (as in Serre's *Local Fields*, Chapter IV).

Let F be a complete discrete nonarchimedean field whose residue field κ_F is *perfect* (this hypothesis is crucial!). Let E be a finite Galois extension of F. The *lower numbering filtration* of $G_{E/F}$ is defined as follows: for $i \geq -1$ an integer.

$$G_{E/F,i} = \ker(G_{E/F} \to \mathrm{Aut}(\mathfrak{o}_F/\mathfrak{m}_F^{i+1})).$$

For $i \geq -1$ real, we define $G_{E/F,i} = G_{E/F,\lceil i \rceil}$. The lower numbering filtration behaves nicely with respect to subgroups of $G_{E/F}$ but not quotients; it thus cannot be defined on the absolute Galois group G_F.

The *upper numbering filtration* of $G_{E/F}$ is defined by the relation $G_{E/F}^{\phi_{E/F}(i)} = G_{E/F,i}$, where

$$\phi_{E/F}(i) = \int_0^i [G_{E/F,0} : G_{E/F,t}]^{-1}\, dt.$$

Note that the indices where the filtration jumps are now rational numbers, but not necessarily integers. In any case, one has the following.

Proposition 4 (Herbrand)**.** *Let E' be a Galois subextension of E/F, and put $H = \mathrm{Gal}(E/E')$, so that H is normal in $G_{E/F}$ and $G_{E/F}/H = G_{E'/F}$. Then $G^i_{E'/F} = (G^i_{E/F}H)/H$; that is, the upper numbering filtration is compatible with forming quotients of $G_{E/F}$.*

Consequently, we obtain a filtration G^i_F on G_F which induces the upper numbering filtration on each $G^i_{E/F}$.

If we take $F = \kappa_K((t))$, we then obtain the following. (The attribution is somewhat complicated, involving Crew, Matsuda, Tsuzuki, Christol-Mebkhout, André, etc.; see the compiled notes.)

Theorem 5. *Assume that κ_K is perfect. Let V be a finite dimensional vector space over K, and let $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ be a continuous homomorphism for the* discrete *topology on $\mathrm{GL}(V)$. Then for $\rho \in (0,1)$ sufficiently close to 1,*

$$R(D^\dagger(V) \otimes F_\rho) = \rho^b, \qquad b = \max\{i \geq 1 : G_{\kappa_K((t)),i} \not\subseteq \ker(\tau)\}.$$

Corollary 6. *Let $V_1, \dots, V_m$ be the constituents of V, and let $\tau_j : G_{\kappa_K((t))} \to \mathrm{GL}(V_j)$ be the corresponding homomorphisms. For $\rho \in (0,1)$ sufficiently close to 1, the multiset of subsidiary radii of $D^\dagger(V) \otimes F_\rho$ consists of* $\max\{i \geq 1 : G_{\kappa_K((t)),i} \not\subseteq \ker(\tau_j)\}$ *with multiplicity* $\dim(V_j)$*, for $j = 1, \dots, m$.*

Using the integrality properties of subsidiary radii, we may deduce that for $\rho \in (0,1)$ sufficiently close to 1, the the product of the subsidiary radii is an integral power of ρ; this amounts to verifying the *Hasse-Arf theorem* for V (integrality of the Artin conductor).

One might reasonably wonder whether there is a good analogue of Theorem 5 in case the residue field of K is not perfect. There are several difficulties, one of which is to decide upon a good analogue of the upper numbering filtration. Such an analogue has been constructed by Abbes and Saito; the resulting analogue of Theorem 5 was proved recently by Chiarellotto and Pulita for $\dim(V) = 1$, and more recently, by Liang Xiao in general.

4 Representations with finite image of inertia

Let $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ be a homomorphism which is now continuous for the p-adic topology on V, rather than the discrete topology. One can form a differential module over $\mathcal{E}$ by taking

$$D(V) = (V \otimes_K \widehat{\mathcal{E}^{\mathrm{unr}}})^{G_{\kappa_K((t))}}$$

but this in general does not descend to $\mathcal{E}^\dagger$.

Suppose, however, that the image of $G_{\kappa_K((t)),1} \cong G_{\kappa_K^{\mathrm{sep}}((t))}$ (the inertia subgroup) is finite; that is, τ has *finite local monodromy.* Let $\mathcal{E}^\dagger_{\kappa_K^{\mathrm{sep}}((t))}$ be the ring defined in the same fashion as $\mathcal{E}^\dagger$ but using $\widehat{K^{\mathrm{unr}}}$ on the coefficients; let $G_{\kappa_K((t))}$ act on this ring via its unramified quotient. We can then define

$$D^\dagger(V) = (V \otimes_K (\mathcal{E}^\dagger_{\kappa_K^{\mathrm{sep}}((t))})^{\mathrm{unr}})^{G_{\kappa_K((t))}}$$

and this will be a differential module over $\mathcal{E}^\dagger$ of the right dimension, again carrying a unit-root Frobenius structure.

5 Unit-root Frobenius structures

If M is a differential module over $\mathcal{E}^\dagger$, we say M is *quasiconstant* if $M \otimes \mathcal{E}^\dagger_L$ admits a basis of horizontal sections for some L.

Theorem 7 (Tsuzuki)**.** *Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure for some Frobenius lift. Then M is quasiconstant.*

It is important to note that the existence of a unit-root Frobenius structure for one Frobenius lift implies the same for any other Frobenius lift. In fact, for the proof we need to make a more precise observation, which is easy to check from the change of Frobenius construction. Let us say that for $c \in [0,1)$, a basis $e_1, \dots, e_n$ of M is *c-constant* if Φ acts on this basis via a matrix $A = \sum_i A_i t^i$ satisfying $|A|_1 = |A^{-1}| = 1$ and $|A - A_0|_1 \leq c$.

Lemma 8. *Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure for some Frobenius lift ϕ_1. Suppose that $e_1, \dots, e_n$ is a c-constant basis. Then $e_1, \dots, e_n$ is also c-constant for the Frobenius structure corresponding to any other Frobenius lift ϕ_2.*

Given this lemma, the proof can be broken into three steps; we refer to Tsuzuki's original paper (*Amer. J. Math.* 1998) for the proofs.

Lemma 9. *Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure. Then there exists a positive integer m coprime to p such that $M \otimes \mathcal{E}^\dagger[t^{1/m}]$ admits a c-constant basis for some $c \in (0,1)$.*

Lemma 10. *Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure and a c-constant basis for some $c \in (0,1)$. Then for some finite extension L of $\kappa_K((t))$, $M \otimes \mathcal{E}^\dagger_L$ admits a c'-constant basis for some $c' \in (0,c)$.*

These first two lemmas are easy for an absolute Frobenius lift, because you can actually choose the basis to be fixed modulo some power of $\mathfrak{m}_K$. This is the only case Tsuzuki originally addressed; however, in the compiled notes, I will explain how to modify the argument slightly to work for a general Frobenius lift.

Lemma 11. *Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure. Suppose that M admits a c-constant basis for some $c < p^{-1/(p-1)}$. Then M is constant.*

As noted by Christol, this last lemma can be proved elegantly using Frobenius antecedents.

6 Quasiunipotent differential modules

Define the *Robba ring* to be

$$\mathcal{R} = \cup_{\alpha \in (0,1)} K\langle \alpha/t, t\}\};$$

that is, $\mathcal{R}$ consists of formal sums $\sum c_i t^i$ which converge in some range $\alpha \leq |t| < 1$, but need not have bounded coefficients. Unlike its subring $\mathcal{E}^\dagger$, $\mathcal{R}$ is not a field; for instance, the element

$$\log(1+t) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} t^i$$

is not invertible (because its Newton polygon has infinitely many slopes). More generally, we have the following easy fact.

Lemma 12. *We have $\mathcal{R}^\times = (\mathcal{E}^\dagger)^\times$.*

In particular, $\mathcal{R}$ does not have a natural p-adic topology. The most useful topology on $\mathcal{R}$ is the *LF topology*, which is the direct limit of the Fréchet topology on each $K\langle \alpha/t, t\}\}$ defined by the $|\cdot|_\rho$ for $\rho \in [\alpha, 1)$.

In fact, the ring $\mathcal{R}$ is not even noetherian (this is related to an earlier exercise), but the following useful facts are true, essentially by work of Lazard. (These depend on K being spherically complete, which follows from our hypothesis that K is in fact discretely valued.)

Proposition 13. *For an ideal I of $\mathcal{R}$, the following are equivalent.*

(a) The ideal I is closed in the LF topology.

(b) The ideal I is finitely generated.

(c) The ideal I is principal.

Proposition 14. *Any finite free module on the half-open annulus with closed inner radius α and open outer radius 1 is represented by a finite free module over $K\langle \alpha/t, t\}\}$, and so corresponds to a finite free module over $\mathcal{R}$. (The first part generalizes to half-open and open annuli with arbitrary boundary radii.)*

For L a finite separable extension of $\kappa_K((t))$, put

$$\mathcal{R}_L = \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger.$$

We say a finite differential module M over $\mathcal{R}$ is *quasiconstant* if there exists L such that $M \otimes \mathcal{R}_L$ is trivial. We say M is *quasiunipotent* if it is a successive extension of quasiconstant modules; it is equivalent to ask that $M \otimes \mathcal{E}_L^\dagger$ be unipotent (i.e., an extension of trivial differential modules) for some L (exercise).

Quasiunipotent differential modules have many useful properties. For instance, by Proposition 3, they are all solvable at 1. Another important property is the following.

Proposition 15. *Let M be a quasiunipotent differential module over $\mathcal{R}$. Then the spaces $H^0(M), H^1(M)$ are finite dimensional, and there is a perfect pairing*

$$H^0(M) \times H^1(M^\vee) \to H^1(M \otimes M^\vee) \to H^1(\mathcal{R}) \cong K\frac{dt}{t}.$$

Proof. This can be reduced to the unipotent case, for which it is an exercise. ☐

The following important theorem asserts that many naturally occurring differential modules, including Picard-Fuchs modules, are quasiunipotent. See the notes for further discussion.

Theorem 16 (p-adic local monodromy theorem)**.** *Let M be a finite differential module over $\mathcal{R}$ admitting a Frobenius structure for some Frobenius lift. Then M is quasiunipotent.*

We will have more to say about this theorem later.

7 Notes

The weak topology on $\mathcal{E}$ is called the *levelwise topology* in [Ked04].

The statement of the p-adic local monodromy theorem (Theorem 16) was originally known under the name *Crew's conjecture*, because it emerged from the work of Crew on finite dimensionality of rigid cohomology with coefficients in an overconvergent F-isocrystal. The original conjecture only concerned modules such that the differential and Frobenius structures were both defined over $\mathcal{E}^\dagger$; this form was restated in a more geometric form by de Jong.

The restricted case of Crew's conjecture just described is the one that appears in applications to p-adic cohomology. However, the general form is in many ways more natural; this was illustrated by the work of Tsuzuki, who explained how for an absolute Frobenius lift, Theorem 16 would follow from a slope filtration theorem. Moreover, the proof by Berger that Crew's conjecture implies Fontaine's conjecture C_{pst} (that de Rham representations are potentially semistable) requires the unrestricted form of Crew's conjecture.

There are essentially two methods for proving Theorem 16, each with its own merits. One method is to follow Tsuzuki's suggestion to construct slope filtrations for difference modules; this was carried out by Kedlaya. (Beware that Tsuzuki's original reduction argument only applies in the case of an absolute Frobenius lift; for the general case, you have to modify it as sketched in these notes.)

The second method is to use various results of Christol-Mebkhout to analyze differential modules which are solvable at 1. This method was carried out by André and Mebkhout (independently of each other and of Kedlaya).

8 Exercises

1. Let M be a differential module over $\mathcal{R}$ such that for some finite separable extension of L, $M \otimes \mathcal{E}^\dagger_L$ is unipotent. Prove that M is quasiunipotent.

2. Prove Proposition 15 in the case where M is unipotent.