p-adic differential equations
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Regular and irregular singularities

In this lecture, we use the theory of Newton polygons for twisted polynomials to give an algebraic treatment of the basic local theory of complex differential equations with meromorphic singularities. Along the way, we develop some results which we will use again in the p-adic setting.

1 Spectral norm of a linear operator

Let F be a field equipped with an absolute value $|\cdot|$, let V be a vector space over F equipped with a compatible absolute value $|\cdot|_V$, and let $T : V \to V$ be a bounded linear transformation. The *operator norm* of T is defined as

$$|T|_V = \sup_{v \in V, v \neq 0} \{|T(v)|/|v|\};$$

the fact that this is finite is precisely the condition that T is bounded.

The operator norm depends strongly on the norm on V (although the property of being bounded only depends on the equivalence class of the norm). A less delicate invariant is the *spectral norm*, defined as

$$|T|_{\mathrm{sp},V} = \lim_{s \to \infty} |T^s|_V^{1/s};$$

the existence of the limit follows from the fact $|T^{m+n}|_V \leq |T^m|_V |T^n|_V$ and the following lemma.

Lemma 1 (Fekete)**.** *Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers such that $a_{m+n} \geq a_m + a_n$ for all m, n. Then the sequence $\{a_n/n\}_{n=1}^\infty$ either converges to its supremum or diverges to $+\infty$.*

Proof. Exercise. □

Proposition 2. *The spectral norm of T depends on the norm $|\cdot|_V$ only up to equivalence.*

Proof. Suppose $|\cdot|'_V$ is an equivalent norm. We can then choose $c > 0$ such that $|v|'_V \leq c|v|_V$ and $|v|_V \leq c|v|'_V$ for all $v \in V$. We then have $|T(v)|_V/|v|_V \leq c^2 |T(v)|'_V/|v|'_V$ for all $v \in V - \{0\}$. Applying this with T replaced by T^s, this gives $|T^s|_V \leq c^2 |T^s|'_V$, so

$$|T^s|_{\mathrm{sp},V} \leq \lim_{s \to \infty} c^{2/s} (|T^s|'_{\mathrm{sp},V})^{1/s}.$$

Since $c^{2/s} \to 1$ as $s \to \infty$, this gives $|T^s|_{\mathrm{sp},V} \leq |T^s|'_{\mathrm{sp},V}$. The reverse inequality holds by reversing the roles of the norms. □

2 Spectral norms for differential operators

Let F be a nonarchimedean differential field. Since d is a bounded operator on F, we can define the spectral norm $|d|_{\mathrm{sp},F}$.

A *normed differential module* over F is a vector space V over F equipped with a norm $|\cdot|_V$ compatible with $|\cdot|_F$, and a derivation D with respect to d which is bounded as an operator on V. (If V is finite dimensional over F, the boundedness condition is automatic; if also F is complete, then any two norms on V are equivalent.) We define the *truncated spectral norm* of D on V as

$$|D|_{\mathrm{tsp},V} = \max\{|d|_{\mathrm{sp},F}, |D|_{\mathrm{sp},V}\}.$$

(I do not know any examples where this differs from $|D|_{\mathrm{sp},V}$ itself, but I cannot prove the equality. See the exercises for a partial result.)

In some cases, it may be useful to compute in terms of a basis of V over F.

Lemma 3. *Suppose that V is finite dimensional over F. Fix a basis $e_1, \dots, e_n$ of V, and let D_s be the matrix via which D^s acts on this basis; that is, $D^s(e_j) = \sum_i (D_s)_{ij} e_i$. Then*

$$|D|_{\mathrm{tsp},V} = \max\{|d|_{\mathrm{sp},F}, \limsup_{s\to\infty} |D_s|^{1/s}\}. \tag{1}$$

I suspect (but have not checked) that if the maximum is only achieved by the second term, then you can replace the limit superior by a limit.

Proof. (Compare [CD94, Proposition 1.3].) Equip V with the supremum norm defined by $e_1, \dots, e_n$; then $|D^s|_V \geq \max_{i,j} |(D_s)_{i,j}|$. This implies that the left side of (1) is greater than or equal to the the right side.

Conversely, for any $x \in V$, if we write $x = x_1 e_1 + \cdots + x_n e_n$, then

$$D^s(x) = \sum_{i=1}^n \sum_{j=0}^s \binom{s}{j} d^j(x_i) D^{s-j}(e_i),$$

so

$$|D^s|_V^{1/s} \leq \max_{0\leq j\leq s} \{|d^j|_F^{1/s} |D_{s-j}|^{1/s}\}. \tag{2}$$

Given $\epsilon > 0$, we can choose $c = c(\epsilon)$ such that for all $s \geq 0$,

$$\begin{aligned} |d^s|_F &\leq c(|d|_{\mathrm{sp},F} + \epsilon)^s \\ |D_s| &\leq c(\limsup_{s\to\infty} |D_s|^{1/s} + \epsilon)^s. \end{aligned}$$

(The c is only needed to cover small s.) Then (2) implies

$$|D^s|_V^{1/s} \leq c^{2/s} \max\{|d|_{\mathrm{sp},F} + \epsilon, \limsup_{s\to\infty} |D_s|^{1/s} + \epsilon\}.$$

As $s \to \infty$, the factor $c^{2/s}$ tends to 1. From this it follows that the right side of (1) is greater than or equal to the left side minus ϵ; since $\epsilon > 0$ was arbitrary, we get the same inequality with $\epsilon = 0$. $\square$

Lemma 4. *(a) For $0 \to V_1 \to V \to V_2 \to 0$ a short exact sequence of differential modules,*

$$|D|_{\mathrm{tsp},V} = \max\{|D|_{\mathrm{tsp},V_1}, |D|_{\mathrm{tsp},V_2}\}.$$

(b) For V_1, V_2 differential modules,

$$|D|_{\mathrm{tsp},V_1 \otimes V_2} \leq \max\{|D|_{\mathrm{tsp},V_1}, |D|_{\mathrm{tsp},V_2}\},$$

with equality when $|D|_{\mathrm{tsp},V_1} \neq |D|_{\mathrm{tsp},V_2}$.

(c) For V a finite differential module,

$$|D|_{\mathrm{tsp},V^\vee} = |D|_{\mathrm{tsp},V}.$$

Proof. Straightforward. (In case you don't find the equality part of (b) so straightforward, I will demonstrate how to deduce it from the other facts in a subsequent unit.) □

Corollary 5. *If V_1, V_2 are irreducible and $|D|_{\mathrm{tsp},V_1} \neq |D|_{\mathrm{tsp},V_2}$, then every irreducible submodule W of $V_1 \otimes V_2$ satisfies $|D|_{\mathrm{tsp},W} = \max\{|D|_{\mathrm{tsp},V_1}, |D|_{\mathrm{tsp},V_2}\}$.*

There might be a simple proof improving this to cover irreducible subquotients of $V_1 \otimes V_2$, but I don't know of one. I'll deduce something slightly weaker later (Corollary 10).

Proof. Suppose the contrary; we may assume that $|D|_{\mathrm{tsp},V_1} > |D|_{\mathrm{tsp},V_2}$. The inclusion $W \hookrightarrow V_1 \otimes V_2$ corresponds to a nonzero horizontal section of $W^\vee \otimes V_1 \otimes V_2 \cong (W \otimes V_2^\vee)^\vee \otimes V_1$, which in turn corresponds to a nonzero map $W \otimes V_2^\vee \to V_1$. Since V_1 is irreducible, the map has image V_1; that is, $W \otimes V_2^\vee$ has a quotient isomorphic to V_1.

However, we can contradict this using Lemma 4. Namely,

$$|D|_{\mathrm{tsp},W \otimes V_2^\vee} \leq \max\{|D|_{\mathrm{tsp},W}, |D|_{\mathrm{tsp},V_2}\} < |D|_{\mathrm{tsp},V_1},$$

so each nonzero subquotient of $W \otimes V_2^\vee$ has truncated spectral norm strictly less than $R(V_1)$. □

3 A coordinate-free approach

I mention in passing the following more coordinate-free approach to defining the truncated spectral norm; in particular, there is no need to explicitly truncate when using this method.

Proposition 6 (Baldassarri-di Vizio)**.** *Let F be a nonarchimedean differential field of characteristic 0 with d nontrivial; put $F_0 = \ker(d)$. Let $F\{T\}^{(s)}$ be the set of twisted polynomials of degree at most s; define the norm of $P \in F\{T\}^{(s)}$ as $|P(d)|_F$ (that is, consider $P(d)$ as an operator on F). Let V be a finite differential module over F, and fix a norm on V compatible with $|\cdot|$. Let $L_{F_0}(V)$ be the space of bounded F_0-linear endomorphisms of V, equipped with the operator norm. Let $D_s : F\{T\}^{(s)} \to L_{F_0}(V)$ be the map $P \mapsto P(D)$. Then*

$$|D|_{\mathrm{tsp},V} = |d|_{\mathrm{sp},F} \lim_{s \to \infty} |D_s|^{1/s}. \tag{3}$$

Proof. We have $|D|_{\mathrm{tsp},V} \leq |d|_{\mathrm{sp},F} \liminf_{s\to\infty} |D_s|^{1/s}$ because on one hand $|D^s|_V \leq |d^s|_F |D_s|$ by taking $T^s \in F\{T\}^{(s)}$, and on the other hand $\liminf |D_s|^{1/s} \geq 1$ because $1 \in F\{T\}^{(n)}$. In the other direction, we may prove $|D|_{\mathrm{tsp},V} \geq |d|_{\mathrm{sp},F} \limsup_{s\to\infty} |D_s|^{1/s}$ by imitating the proof of Lemma 3. ☐

4 Twisted polynomials and spectral norms

When F is a nonarchimedean differential field, we have been writing

$$r_0 = \min_{f\in F}\{v(d(f)) - v(f)\}.$$

In our new notation, this is just $-\log |d|_F$.

Theorem 7 (Christol-Dwork). *Let F be a complete nonarchimedean differential field, and let V be a finite dimensional differential module over F. Assume that V admits a cyclic vector, and write $V \cong F\{T\}/F\{T\}P$ for some $P \in F\{T\}$. Let r be the least slope of the Newton polygon of P. Then*

$$\max\{|d|_F, |D|_{\mathrm{tsp},V}\} = \max\{|d|_F, e^{-r}\}.$$

You might want to ponder the case of d trivial first, as the general case is similar.

Proof. By factoring P as in the previous unit, then applying Lemma 4, we may reduce to the case where either P has a single slope $r < r_0$, or P has all slopes at least r_0.

Write $P = T^d + \sum_{i=0}^{d-1} P_i T^i$, so that $|P_i| \leq e^{-r(d-i)}$ with equality for $i = 0$. Equip V with the norm

$$|a_0 + a_1 T + \cdots + a_{d-1}T^{d-1}|_V = \max_i\{|a_i| e^{-ri}\}.$$

Let U be the F-linear map defined by

$$U(T^i) = T^{i+1} \quad (i = 0, \dots, d-2), \qquad U(T^{d-1}) = -\sum_{i=0}^{d-1} P_i T^i.$$

Suppose that $r \leq r_0$ and that P has all slopes at least r. Then $|U^n|_V \leq e^{-rn}$ for all nonnegative integers n. On the other hand, $|D - U|_V \leq |d|_F$, so $|D^n - U^n|_V \leq |d|_F e^{-r(n-1)}$. Hence $|D^n|_V \leq e^{-rn}$.

In case P has all slopes at least r_0, we take $r = r_0$ above and deduce the desired result. In case P has all slopes equal to r, we have $|U^n|_V = e^{-rn} > |D^n - U^n|_V$ and so $|D^n|_V = e^{-rn}$, again as desired. ☐

5 Decomposition by truncated spectral norm

Theorem 8 (Weak decomposition theorem)**.** *Let F be a complete nonarchimedean differential field of characteristic zero with nontrivial derivation, and let V be a finite dimensional differential module over F. Then there exists a decomposition*

$$V = V_0 \oplus \bigoplus_{s > |d|_F} V_s$$

of differential modules, such that every subquotient of V_s has truncated spectral norm s, and every subquotient of V_0 has truncated spectral norm at most $|d|_F$.

Proof. We induct on $\dim(V)$. Choose a cyclic vector for V (possible because of the hypotheses we imposed on F), yielding an isomorphism $V \cong F\{T\}/F\{T\}P$. Let r be the least slope of P. If $r \geq r_0$, we may put $V = V_0$ and be done, so assume $r < r_0$. Factor P by slopes as in the previous unit; this gives a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ in which (by Theorem 7) every subquotient of V_1 has truncated spectral norm e^{-r}, and every subquotient of V_2 has truncated spectral norm less than e^{-r}. Factoring P the other way, we get a short exact sequence $0 \to V_2' \to V \to V_1' \to 0$ where every subquotient of V_1' has truncated spectral norm e^{-r}, and every subquotient of V_2' has truncated spectral norm less than e^{-r}. Moreover, $\dim V_1 = \dim V_1'$ and $\dim V_2 = \dim V_2'$ because P and its formal adjoint have the same multiplicities for slopes less than r_0. Consequently, $V_1 \cap V_2' = 0$, so $V_1 \oplus V_2'$ injects into V; by counting dimensions, this must be an isomorphism. This lets us split $V \cong V_1 \oplus V_2$, and we may apply the induction hypothesis to V_2 to get what we want. □

Corollary 9. *Let F be a complete nonarchimedean differential field, and let V be a finite dimensional differential module over F such that every subquotient of V has truncated spectral norm greater than $|d|_F$. Then $H^0(V) = H^1(V) = 0$.*

Proof. The claim about H^0 is clear: a nonzero element of $H^0(V)$ would generate a differential submodule of V which would be trivial, and thus would have truncated spectral norm $|d|_{\mathrm{sp},F} \leq |d|_F$. As for H^1, let $0 \to V \to W \to F \to 0$ be a short exact sequence of differential modules. Decompose $W = W_0 \oplus W_1$ according to Theorem 8, with every subquotient of W_0 having truncated spectral norm at most $|d|_F$, and every subquotient of W_1 having truncated spectral norm greater than $|d|_F$. The map $V \to W_0$ must vanish (its image is a subquotient of both V and W_0), so $V \subseteq W_1$. But $W_1 \neq W$ as otherwise W could not surject onto a trivial module, so $V = W_1$. Hence the sequence splits, proving $H^1(V) = 0$. □

Corollary 10. *If V_1, V_2 are irreducible, $|D|_{\mathrm{tsp},V_1} > |d|_F$, and $|D|_{\mathrm{tsp},V_1} > |D|_{\mathrm{tsp},V_2}$, then every irreducible subquotient W of $V_1 \otimes V_2$ satisfies $|D|_{\mathrm{tsp},W} = |D|_{\mathrm{tsp},V_1}$.*

Proof. Decompose $V_1 \otimes V_2 = V_0 \oplus \bigoplus_{s > |d|_F} V_s$ according to Theorem 8; we have $V_s = 0$ whenever $s > |D|_{\mathrm{tsp},V_1}$. If either V_0 or some V_s with $s < |D|_{\mathrm{tsp},V_1}$ were nonzero, then $V_1 \otimes V_2$ would have an irreducible submodule of spectral norm less than $|D|_{\mathrm{tsp},V_1}$, in violation of Corollary 5. □

For the study of irregularity, these results are quite sufficient. However, in the p-adic situation, we will have to do better than this in order to further decompose V_0; we will do this using Frobenius antecedents in a later unit.

6 Irregularity

We now take $F = \mathbb{C}((z))$, $|\cdot| = e^{-v_z(\cdot)}$, and $d = z\frac{d}{dz}$, so that $|d|_F = 1$ and $r_0 = 0$. That is, the Newton polygon theory for twisted polynomials over F applies for negative slopes. Moreover, $|d|_{\mathrm{sp},F} = 1$ also, so there is no shortfall in Theorem 8; the term V_0 has truncated spectral norm 1.

Let V be a finite differential module over F. Decompose V according to Theorem 8. The *irregularity* of V is defined as

$$\mathrm{irr}(V) = \sum_{s>1} (-\log s)\dim(V_s).$$

By our previous results, we can now read off the following result.

Theorem 11. *For any isomorphism $V \cong F\{T\}/F\{T\}P$, the irregularity of P is equal to the sum of the negative slopes of P; consequently, it is always an integer. More explicitly, if $P = T^d + \sum_{i=0}^{d-1} P_i T^i$, then*

$$\mathrm{irr}(V) = \max_i \{-v_z(P_i)\}.$$

For F a subfield of $\mathbb{C}((z))$ stable under d, and V a finite differential module over F, we define the irregularity by extending scalars to $\mathbb{C}((z))$.

Corollary 12. *Let F be any subfield of $\mathbb{C}((z))$ containing z, and let V be a finite-dimensional differential module over F. Then the following are equivalent.*

(a) The irregularity of V is equal to 0.

(b) For some isomorphism $V \cong F\{T\}/F\{T\}P$ with P monic, P has coefficients in $\mathfrak{o}_F$.

(c) For any *isomorphism $V \cong F\{T\}/F\{T\}P$ with P monic, P has coefficients in $\mathfrak{o}_F$.*

(d) There exists a basis of V on which D acts via a matrix over $\mathfrak{o}_F$.

Proof. It is clear that (a) implies (c) implies (b) implies (d). Given (d), let $|\cdot|_V$ be the supremum norm defined by the chosen basis of V; then $|D|_V \leq 1$, so $|D|_{\mathrm{sp}} \leq 1$, which implies (a). $\square$

We say that V is *regular* if any of the equivalent conditions hold.

7 More on regular singularities

Let us take a moment to see how this algebraic setup informs the study of differential equations over $\mathbb{C}$ with meromorphic singularities; as this is a bit tangential for our purposes, we defer to [DGS94] for most details.

We first record something you already knew.

Theorem 13. *Fix $\rho > 0$, and let $R \subset \mathbb{C}[\![z]\!]$ be the ring of power series convergent for $|z| < \rho$. Let N be an $n \times n$ matrix over R congruent to 0 modulo z. Then the differential system $D(v) = Nv + d(v)$ has a basis of horizontal sections.*

Proof. Apply the fundamental theorem of ordinary differential equations. □

Let $\mathbb{C}\{z\}$ be the subring of $\mathbb{C}((z))$ consisting of the Laurent series expansions of functions meromorphic in a neighborhood of $z = 0$; then $\mathbb{C}\{z\}$ satisfies the hypothesis of Corollary 12. Consider the differential system $D(v) = Nv + d(v)$, for N an $n \times n$ matrix over $\mathbb{C}\{z\}$. We can find a basis of horizontal sections on a disc around some point near 0 (by the previous theorem, it suffices to take a disc on which the entries of N are all holomorphic away from $z = 0$), then analytically continue them around to get a different basis of the same space. The linear transformation affected by the analytic continuation is called the *monodromy transformation* of the system, and we would dearly like to get a closer look at it, for instance, because of the following observation.

Proposition 14. *Any fixed vector under the monodromy transformation corresponds to a horizontal section defined on some punctured disc, rather than the universal covering space of a punctured disc. As a result, the monodromy transformation is unipotent if and only if there exists a basis on which D acts via a nilpotent matrix.*

Proof. Straightforward. □

It is rather difficult in general to get information about the monodromy transformation by looking at the differential system. What makes regular singularities so pleasant is that it is quite easy to read off the eigenvalues of the monodromy transformation; this is analogous to the fact that the residue of a complex-valued function is quite easy to read off at a simple pole and somewhat trickier to compute otherwise.

We say that a square matrix N has *prepared eigenvalues* if the eigenvalues $\lambda_1, \dots, \lambda_n$ of N satisfy the following conditions:

$$\lambda_i \in \mathbb{Z} \Leftrightarrow \lambda_i = 0$$
$$\lambda_i - \lambda_j \in \mathbb{Z} \Leftrightarrow \lambda_i = \lambda_j.$$

If only the second condition holds, we say that N has *weakly prepared eigenvalues*.

Theorem 15 (Fuchs)**.** *Let V be a regular finite dimensional differential module over $\mathbb{C}\{z\}$, and let $e_1, \dots, e_n$ be a basis on which D acts via a matrix N over the valuation ring (the ring of power series over $\mathbb{C}$ with positive radii of convergence). Assume that the matrix N_0 of constant terms of N has weakly prepared eigenvalues. Then there is another basis of V on which D acts via N_0.*

Proof. See exercises, or [DGS94, §III.8, Appendix II]. ☐

Corollary 16. *With notation as in Theorem 15, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of N_0. Then the eigenvalues of the monodromy transformation (of the system $D(v) = Nv + dv$) are $e^{-2\pi i\lambda_1}, \dots, e^{-2\pi i\lambda_n}$.*

Proof. In terms of a basis via which D acts via N_0, the matrix $\exp^{-N_0 \log(z)}$ provides a basis of horizontal elements. (The case $N_0 = 0$ is Theorem 13.) ☐

In order to enforce the condition on prepared eigenvalues, we use what are classically known as *shearing transformations*.

Proposition 17 (Shearing transformations)**.** *Let N be an $n \times n$ matrix over the valuation ring of $\mathbb{C}\{z\}$, with constant term N_0. Let α be an eigenvalue of N. Then there exists $U \in \mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ such that $U^{-1}NU + U^{-1}d(U)$ again has entries in the valuation ring of $\mathbb{C}\{z\}$, and its matrix of constant terms has the same eigenvalues as N_0 except that α has been replaced by $\alpha + 1$. The same conclusion holds with $\alpha - 1$ in place of $\alpha + 1$.*

Proof. Exercise. ☐

Corollary 18 (Fuchs)**.** *Let V be a regular finite dimensional differential module over $\mathbb{C}\{z\}$. Then any horizontal element of $V \otimes \mathbb{C}((z))$ belongs to V itself; that is, any formal horizontal section is convergent.*

In particular, the eigenvalues of N modulo z are well determined modulo $\mathbb{Z}$; they are called the *exponents* of the differential module. A particularly important case is when the exponents are all rational, as this implies that the module is *quasi-unipotent*, i.e., after pulling back along $z \mapsto z^m$ for some positive integer m, the module becomes a successive extension of trivial differential modules. Quasi-unipotent monodromy occurs whenever V "comes from geometry" in a sense that we will make precise later.

8 Index and irregularity

Let F be any subfield of $\mathbb{C}((z))$ containing $\mathbb{C}(z)$, and let V be a finite differential module over F. We say V *has index* if $\dim_{\mathbb{C}} H^0(V)$ and $\dim_{\mathbb{C}} H^1(V)$ are both finite; in this case, we define the *index* of V as $\chi(V) = \dim_{\mathbb{C}} H^0(V) - \dim_{\mathbb{C}} H^1(V)$.

Proposition 19. *For any finite differential module V over $\mathbb{C}((z))$, $H^0(V) = H^1(V) = 0$.*

Proof. Exercise. ☐

In the convergent case, the index carries more information.

Theorem 20. *Let V be a finite differential module over $\mathbb{C}\{z\}$. Then V has index, and $\chi(V) = -\operatorname{irr}(V)$.*

Proof. See for instance [Mal74, Théorème 2.1]. ☐

9 Notes

Proposition 6 is from as yet unreleased work of Baldassarri and di Vizio (a promised sequel to [BdV07]), which gives a development of much of the material we are discussing from the point of view of Berkovich analytic spaces. This point of view will probably be vital for the study of differential modules on higher-dimensional spaces, but we will not be doing that in these notes.

The proof of Theorem 7 given originally in [CD94, Théorème 1.5] is slightly incorrect. The error is in the implication $1 \implies 2$; there one makes a finite extension of the differential field, without accounting for the possibility that this might increase $|d|_F$. We get around this by using Robba's factorization argument; otherwise, the proof above is very similar to that of Christol and Dwork. A similar argument appears as [DGS94, Lemma VI.2.1], but in the language of generic radii of convergence which we will introduce in the next unit.

The notion a regular singularity was introduced by Fuchs in the 19th century, as part of a classification of those differential equations with everywhere meromorphic singularities on the Riemann sphere which had algebraic solutions. Regular singularities are sometimes referred to as *Fuchsian singularities*. Much of our modern understanding of the regularity condition, especially in higher dimensions, comes from the book of Deligne [Del70].

The algebraic definition of irregularity is due to Malgrange [Mal74]; it had previously been defined in terms of the index of a certain operator. Our approach, incorporating ideas of Robba, is based on [DGS94, §3].

A complex analytic interpretation of the Newton polygon, in the manner of the relation between irregularity and index, has been given by Ramis [Ram84]. It involves considering subrings of $\mathbb{C}\{z\}$ composed of functions with certain extra convergence restrictions (*Gevrey functions*), and looking at the index of $z\,d/dz$ after tensoring the given differential module with one of these subrings.

10 Exercises

1. Prove Fekete's lemma (Lemma 1).

2. Let A, B be commuting bounded linear operators on a normed vector space V over a nonarchimedean field F. Prove that

$$|A + B|_{\mathrm{sp},V} \leq \max\{|A|_{\mathrm{sp},V}, |B|_{\mathrm{sp},V}\},$$

   and that equality occurs when the maximum is achieved only once.

3. Let V be a normed differential module over a normed differential field F, and pick a norm $|\cdot|_V$ on V compatible with $|\cdot|$. Prove that $|D|_V \geq |d|_F$.

4. In this exercise, we prove Fuchs's theorem (Theorem 15). Let N be an $n \times n$ matrix over $\mathbb{C}[\![z]\!]$. Let U be an $n \times n$ matrix over $\mathbb{C}[\![z]\!]$ congruent to the identity modulo z.

(a) Show that changing basis by U in the differential system $D(v) = Nv + d(v)$ has the effect of replacing N by $N' = U^{-1}NU + U^{-1}z\frac{dU}{dz}$.

(b) Show that $N' \equiv N \pmod{z}$.

(c) Assume that the reduction of N modulo z has prepared eigenvalues. Show that there is a unique choice of U for which N' equals the matrix of constant terms of N.

(d) Suppose that the entries of N converge in the disc $|z| < \rho$. Prove that the entries of the matrix U given in (c) also converge in the disc $|z| < \rho$.

5. Prove Proposition 17.

6. Prove Proposition 19.