Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
Adeles and Ideles in Field Extensions

Reference. Neukirch, Section VI.1 and VI.2.

Adeles in Field Extensions.

If L/K is an extension of number fields, we get an embedding $\mathbb{A}_K \hookrightarrow \mathbb{A}_L$ as follows: given $\alpha \in \mathbb{A}_K$, each place w of L restricts to a place v of K, so set the w-component of the image of α to α_v. This embedding induces an inclusion $I_K \hookrightarrow I_L$ of idele groups as well.

If L/K is Galois with Galois group G, then G acts naturally on $\mathbb{A}_L$ and I_L; more generally, if $g \in \mathrm{Gal}(\overline{K}/K)$, then g maps L to some other extension L^g of K, and G induces isomorphisms of $\mathbb{A}_L$ with $\mathbb{A}_{L^g}$ and of I_L with I_{L^g}. Namely, if $(\alpha_v)_v$ is an idele over L and $g \in G$, then g maps the completion L_v of L to a completion L_{v^g} of L^g. (Remember, a place v corresponds to an absolute value $|\cdot|_v$ on L; the absolute value $|\cdot|_{v^g}$ on L^g is given by $|a^g|_{v^g}| = |a|_v$.) As you might expect, this action is compatible with the embeddings of L in I_L and L^g in I_{L^g}, so it induces an isomorphism $C_L \to C_{L^g}$ of idele class groups.

Aside. Neukirch points out that you can regard $\mathbb{A}_L$ as the tensor product $\mathbb{A}_K \otimes_K L$; in particular, this is a good way to see the Galois action on $\mathbb{A}_L$. Details are left to the reader.

We can define trace and norm maps as well:

$$\mathrm{Trace}_{\mathbb{A}_L/\mathbb{A}_K}(x) = \sum_g x^g, \qquad \mathrm{Norm}_{I_L/I_K}(x) = \prod_g x^g$$

where g runs over coset representatives of $\mathrm{Gal}(\overline{K}/L)$ in $\mathrm{Gal}(\overline{K}/K)$, the sum and product taking places in the adele and idele rings of the Galois closure of L over K. In particular, if L/K is Galois, g simply runs over $\mathrm{Gal}(L/K)$.

In terms of components, these definitions translate as

$$\begin{aligned}(\mathrm{Trace}_{\mathbb{A}_L/\mathbb{A}_K}(\alpha))_v &= \sum_{w|v} \mathrm{Trace}_{L_w/K_v}(\alpha_w)\\ (\mathrm{Norm}_{I_L/I_K}(\alpha))_v &= \prod_{w|v} \mathrm{Norm}_{L_w/K_v}(\alpha_w).\end{aligned}$$

The trace and norm do what you expect on principal adeles/ideles. In particular, the norm descends to a map $\mathrm{Norm}_{L/K} : C_L \to C_K$.

Aside. You can also define the trace of an adele $\alpha \in \mathbb{A}_L$ as the trace of addition by α as an endomorphism of the $\mathbb{A}_K$-module $\mathbb{A}_L$, and the norm of an idele $\alpha \in I_L$ as the determinant of multiplication by α as an automorphism of the $\mathbb{A}_K$-module $\mathbb{A}_L$. (Yes, the action is on the *adeles* in both cases. Remember, ideles should be thought of as automorphisms of the adeles, not as elements of the adele ring.)

If L/K is a Galois extension, then $\mathrm{Gal}(L/K)$ acts on $\mathbb{A}_L$ and I_L fixing $\mathbb{A}_K$ and I_K, respectively, and we have the following.

Proposition 1. *If L/K is a Galois extension with Galois group G, then $\mathbb{A}_L^G = \mathbb{A}_K$ and $I_L^G = I_K$.*

Proof. If v is a place of K, then for each place w of K above v, the decomposition group G_w of w is isomorphic to $\mathrm{Gal}(L_w/K_v)$. So if (α) is an adele or idele which is G-invariant, then α_w is $\mathrm{Gal}(L_w/K_v)$-invariant for each w, so belongs to K_v. Moreover, G acts transitively on the places w above v, so $\alpha_w = \alpha_{w'}$ for any two places w, w' above v. Thus (α) is an adele or idele over K. □

This has the following nice consequence for the idele class group, a fact which is quite definitely not true for the ideal class group: the map $\mathrm{Cl}_K \to \mathrm{Cl}_L^G$ is neither injective nor surjective in general. This is our first hint of why the idele class group will be a more convenient target for a reciprocity map than the ideal class group.

Proposition 2 (Galois descent). *If L/K is a Galois extension with Galois group G, then G acts on C_L, and the G-invariant elements are precisely C_K.*

Proof. We start with an exact sequence

$$1 to L^* \to I_L \to C_L \to 1$$

of G-modules. Taking G-invariants, we get a long exact sequence

$$1 \to (L^*)^G = K^* \to (I_L)^G = I_K \to C_L^G \to H^1(G, L^*),$$

and the last term is 1 by (Hilbert-Noether) Theorem 90. So we again have a short exact sequence, and $C_L^G \cong I_K/K^* = C_K$. □