Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
Adeles and Ideles

Reference. Milne, Section V.4; Neukirch, Section VI.1 and VI.2; Lang, Chapter VII.

Last semester, we introduced the p-adic numbers and more general local fields as a way to translate "local" facts about number fields into statements of a "topological" flavor. This semester, we need to do something analogous for "global" facts, such as the statements of class field theory. To do this, we construct a topological object that includes all of the completions of a number field, including both the archimedean and nonarchimedean ones. This object will be the ring of adeles, and it will lead us to the right target group for use in the abstract class field theory we have just set up.

Spelling note. The words "adele" and "idele" are sometimes spelled "adèle" and "idèle" (as in Neukirch), but not as far as I know by anyone who speaks French (e.g., Lang).

Jargon watch. By a *place* of a number field K, we mean either an archimedean completion $K \hookrightarrow \mathbb{R}$ or $K \hookrightarrow \mathbb{C}$ (an *infinite place*), or a $\mathfrak{p}$-adic completion $K \hookrightarrow K_{\mathfrak{p}}$ for some nonzero prime ideal $\mathfrak{p}$ of $\mathfrak{o}_K$ (a *finite place*). (Note: there is only one place for each pair of complex embeddings of K.) Each place corresponds to an equivalence class of absolute values on K; if v is a place, we write K_v for the corresponding completion, which is either $\mathbb{R}$, $\mathbb{C}$, or $K_{\mathfrak{p}}$ for some prime $\mathfrak{p}$.

The adeles.

The basic idea is that we want some sort of "global completion" of a number field K. In fact, we already know one way to complete $\mathbb{Z}$, namely its profinite completion $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. But we really want something containing $\mathbb{Q}$. We define the *ring of finite adeles* $\mathbb{A}_{\mathbb{Q}}^{\text{fin}}$ as any of the following isomorphic objects:

- the tensor product $\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$;
- the direct limit of $\frac{1}{n}\widehat{\mathbb{Z}}$ over all nonzero integers n;
- the *restricted direct product* $\prod_p{}' \mathbb{Q}_p$, where we only allow tuples (α_p) for which $\alpha_p \in \mathbb{Z}_p$ for almost all p.

For symmetry, we really should allow *all* places, not just the finite places. So we also define the *ring of adeles* over $\mathbb{Q}$ as $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_{\mathbb{Q}}^{\text{fin}}$. Then $\mathbb{A}_{\mathbb{Q}}$ is a locally compact topological ring with a canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$.

Now for a general number field K. The profinite completion $\widehat{\mathfrak{o}_K}$ is canonically isomorphic to $\prod_{\mathfrak{p}} \mathfrak{o}_{K_{\mathfrak{p}}}$, so we define the *ring of finite adeles* $\mathbb{A}_K^{\text{fin}}$ as any of the following isomorphic objects:

- the tensor product $\widehat{\mathfrak{o}_K} \otimes_{\mathfrak{o}_K} K$;
- the direct limit of $\frac{1}{\alpha}\widehat{\mathfrak{o}_K}$ over all nonzero $\alpha \in \mathfrak{o}_K$;
- the *restricted direct product* $\prod_{\mathfrak{p}}{}' \mathbb{Q}_{\mathfrak{p}}$, where we only allow tuples (α_p) for which $\alpha_p \in \mathbb{Z}_p$ for almost all p.

The ring of adeles $\mathbb{A}_K$ is the product of $\mathbb{A}_K^{\text{fin}}$ with each archimedean completion. (That's one copy of $\mathbb{R}$ for each real embedding and one copy of $\mathbb{C}$ for each conjugate pair of complex embeddings.)

One has a natural norm on the ring of adeles, because one has a natural norm on each completion:

$$|(\alpha_v)_v| = \prod_v |\alpha_v|_v.$$

One should normalize these in the following way: for v real, take $|\cdot|_v$ to be the usual absolute value. For v complex, take $|\cdot|_v$ to be the *square* of the usual absolute value. (That means the result is not an absolute value, in that it doesn't satisfy the triangle inequality. Sorry.) For v nonarchimedean corresponding to a prime above p, normalize so that $|p|_v = p^{-1}$.

Again, there is a natural embedding of K into $\mathbb{A}_K$ because there is such an embedding for each completion. With the normalization as above, one has the product formula:

Proposition 1. *If $\alpha \in K$, then $|\alpha| = 1$.*

In particular, K is *discrete* in $\mathbb{A}_K$ (the difference between two elements of K cannot be simultaneously small in all embeddings). This is a generalization/analogue of the fact that $\mathfrak{o}_K$ is discrete in Minkowski space (the product of the archimedean completions).

For any finite set S of places, let $\mathbb{A}_S$ (resp. $\mathbb{A}_S^{\text{fin}}$ be the subring of $\mathbb{A}_K$ (resp. $\mathbb{A}_K^{\text{fin}}$) consisting of those adeles which are integral at all finite places not contained in S. Then we have the following result, which is essentially the Chinese remainder theorem.

Proposition 2. *For any finite set S of places, $K + \mathbb{A}_S^{\text{fin}} = \mathbb{A}_K^{\text{fin}}$ and $K + \mathbb{A}_S = \mathbb{A}_K$.*

Corollary 3. *The quotient group $\mathbb{A}_K/K$ is compact.*

Proof. Choose a compact subset T of Minkowski space M (the product of the archimedean completions) containing a fundamental domain for the lattice $\mathfrak{o}_K$. Then every element of $M \times \mathbb{A}^{\text{fin}}$ is congruent modulo $\mathfrak{o}_K$ to an element of $T \times \mathbb{A}^{\text{fin}}$. By the proposition, the compact set $T \times \mathbb{A}^{\text{fin}}$ surjects onto $\mathbb{A}_K/K$, so the latter is also compact. $\square$

Alternate description: restricted products of topological groups.

Let $G_1, G_2, \ldots$ be a sequence of locally compact topological groups and let H_i be a compact subgroup of G_i. The *restricted product* G of the pairs (G_i, H_i) is the set of tuples $(g_i)_{i=1}^{\infty}$ such that $g_i \in H_i$ for all but finitely many indices i. For each set S, this product contains the subgroup G_S of tuples (g_i) such that $g_i \in H_i$ for $i \notin S$, and indeed G is the direct limit of the G_S. We make G into a topological group by giving each G_S the product topology and saying that $U \subset G$ is open if its intersection with each G_S is open there.

In this language, the additive group of adeles over $\mathbb{Q}$ is simply the restricted product of the pairs $(\mathbb{R}, \mathbb{R})$ and $(\mathbb{Q}_p, \mathbb{Z}_p)$ for each p, and likewise over a number field.

Ideles and the idele class group.

An *idele* is a unit in the ring $\mathbb{A}_K$. In other words, it is a tuple (α_v), one element of K_v^* for each place v of K, such that $\alpha_v \in \mathfrak{o}_{K_v}^*$ for all but finitely many finite places v. Let

I_K denote the group of ideles of K (sometimes thought of as $\mathrm{GL}_1(\mathbb{A}_K)$). This group is the restricted product of the pairs $(\mathbb{R}^*, \mathbb{R}^*)$, $(\mathbb{C}^*, \mathbb{C}^*)$, and $(K_{\mathfrak{p}}^*, \mathfrak{o}_{\mathfrak{p}})^*)$.

For example, for each element $\beta \in K$, we get a principal adele in which $\alpha_v = \beta$; this adele is an idele if $\beta \neq 0$. We call these the *principal adeles* and *principal ideles*, and define the *idele class group* of K as the quotient $C_K = I_K/K^*$ of the ideles by the principal ideles.

Warning. The restricted product topology does not coincide with the topology induced on the ideles by sticking them into the adeles! For example, the set of ideles whose components at finite places are all integral is not an open subset in the ideles, even though it is the intersection of the idele group with an open subset of the adeles.

There is a homomorphism from I_K to the group of fractional ideals of K:

$$(\alpha_\nu)_\nu \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\alpha_{\mathfrak{p}})},$$

which is continuous for the discrete topology on the group of fractional ideals. The principal idele corresponding to $\alpha \in K$ maps to the principal ideal generated by α. Thus we have a surjection $C_K \to \mathrm{Cl}(K)$.

Since the norm is trivial on K^*, we get a well-defined norm map $|\cdot| : C_K \to \mathbb{R}_+^*$. Let C_K^0 be the kernel of the norm map; then C_K^0 also surjects onto $\mathrm{Cl}(K)$. (The surjection onto $\mathrm{Cl}(K)$ ignores the infinite places, so you can adjust there to force norm 1.)

Proposition 4. *The group C_K^0 is compact.*

This innocuous-looking fact actually implies two basic facts from last semester:

(a) The class group of K is finite.

(b) The group of units of K has rank $r+s-1$, where r and s are the number of real and complex places, respectively. More generally, if S is a finite set of places containing the archimedean places, the group of S-units of K (elements of K which have valuation 0 at each finite place not contained in S) has rank $\#(S)-1$.

In fact, (a) is immediate; C_K^0 is compact and it surjects onto $\mathrm{Cl}(K)$, so the latter must also be compact for the discrete topology, i.e., it must be finite. (In fact, $\mathrm{Cl}(K)$ is isomorphic to the group of connected components of C_K^0.) To see (b), let I_S be the group of ideles which are units outside S, and define the map $\log : I_S \to \mathbb{R}^{\#(S)}$ by taking log of the absolute value of the norm of each component in S. By the product formula, this maps into the sum-of-coefficients-zero hyperplane H in $\mathbb{R}^{\#(S)}$, and the image of the group K_S^* of S-units is discrete therein. (Restricting an element of K_S^* to a bounded subset of H bounds all of its absolute values, so this follows from the discreteness of K in $\mathbb{A}_K$.) Let W be the span in H of the image of K_S^*; then we get a continuous homomorphism $C_K^0 \to H/W$ whose image generates H/W. But its image is compact; this is a contradiction unless H/W is the zero vector space. Thus K_S^* must be a lattice in H, so it has rank $\dim H = \#(S)-1$.

Proof of the proposition. The inverse images of any two positive real numbers under the norm map are homeomorphic. So rather than prove that the inverse image of 0 is compact, we'll prove that the inverse image of some $\rho > 0$ is compact. Namely, we choose $\rho > c$, where c has the property that any idele of norm $\rho > c$ is congruent modulo K^* to an idele whose components all have norms in $[1, \rho]$. (Proof that c exists left as an exercise, or see Lang, Section V.1, Theorem 0.)

The set of ideles with each component having norm in $[1, \rho]$ is the product of "annuli" in the archimedean places and finitely many of the nonarchimedean places, and the group of units in the rest. (Most of the nonarchimedean places don't have any valuations between 1 and ρ.) This is a compact set, the set of ideles therein of norm ρ is a closed subset and so is also compact, and the latter set surjects onto C_K^0, so that's compact too. $\square$

One more comment worth making: what are the open subgroups of I_K? In fact, for each formal product $\mathfrak{m}$ of places, one gets an open subgroup of ideles $(\alpha_v)_v$ such that

(a) if v is a real place occurring in $\mathfrak{m}$, then $\alpha_v > 0$;

(b) if v is a finite place corresponding to the prime $\mathfrak{p}$, occurring to the power e, then $\alpha_v \equiv 1 \pmod{\mathfrak{p}^e}$.

And in fact, every open subgroup contains one of these. Thus using the surjection $C_K \mapsto \mathrm{Cl}(K)$, we get a bijection between open subgroups of C_K and generalized ideal class groups!

A presentation of $\mathbb{A}_{\mathbb{Q}}$. In the special case $K = \mathbb{Q}$, the idele class group has a nice presentation. Namely, given an arbitrary idele in $I_{\mathbb{Q}}$, there is a unique positive rational with the same norms at the finite places. Thus

$$I_{\mathbb{Q}} \cong \mathbb{R}^* \times \prod_p \mathbb{Z}_p^*.$$

This definitely does not generalize: as noted above, the idele class group has multiple connected components when the class number is bigger than 1.

Aside: beyond class field theory. You can think of the idele group as $\mathrm{GL}_1(\mathbb{A}_K)$. In that case, class field theory will become a correspondence between one-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ and certain representations of $\mathrm{GL}_1(\mathbb{A}_K)$. This is the form in which class field theory generalizes to the nonabelian case: the Langlands program predicts a correspondence between n-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ and certain representations of $\mathrm{GL}_n(\mathbb{A}_K)$. In fact, with K replaced by the function field of a curve over a finite field, this prediction is a deep theorem of L. Lafforgue (based on work of Drinfeld).