Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
The Adelic Reciprocity Law and Artin Reciprocity

We now describe the setup by which we create a reciprocity law in global class field theory, imitating the "abstract" setup from local class field theory but using the idele class group in place of the multiplicative group of the field. We then work out why the reciprocity law and existence theorem in the adelic setup imply Artin reciprocity and the existence theorem (and a bit more) in the classical language.

Convention note. We are going to fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and regard "number fields" as finite subextensions of $\overline{\mathbb{Q}}/\mathbb{Q}$. That is, we are fixing the embeddings of number fields into $\overline{\mathbb{Q}}$. We'll use these embeddings to decide how to embed one number field in another.

The adelic reciprocity law and existence theorem.

Here are the adelic reciprocity law and existence theorem; notice that they look just like the local case except that the multiplicative group has been replaced by the idele class group.

Theorem 1 (Adelic reciprocity law). *There is a canonical map $r_K : C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ which induces, for each finite extension L/K of number fields, an isomorphism $r_{L/K}$: $C_K/\operatorname{Norm}_{L/K} C_L \to \mathrm{Gal}(L/K)^{\mathrm{ab}}$.*

Theorem 2 (Adelic existence theorem). *For every number field K and every open subgroup H of C_K of finite index, there exists a finite (abelian) extension L of K such that $H = \operatorname{Norm}_{L/K} C_L$.*

In fact, using local class field theory, we can construct the map that will end up being r_K. For starters, let L/K be a finite abelian extension and v a place of K. Put $G = \mathrm{Gal}(L/K)$, and let G_v be the decomposition group of v, that is, the set of $g \in G$ such that $v^g = v$. Then for any place w above v, $G_v \cong \mathrm{Gal}(L_w/K_v)$, so we can view the local reciprocity map $K_v^* \to \mathrm{Gal}(L_w/K_v)$ as a map $r_{K,v} : K_v^* \to G$. That is, if v is a finite place. If $v = \mathbb{C}$, then $\mathrm{Gal}(L_w/K_v)$ is trivial, so we just take $K_v^* \to G$ to be the identity map. If $v = \mathbb{R}$, then we take $K_v^* = \mathbb{R}^* \to \mathrm{Gal}(L_w/K_v) = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ to be the map sending everything positive to the identity, and everything negative to complex conjugation.

Now note that

$$(\alpha_v) \mapsto \prod_v r_{K,v}(\alpha_v)$$

is well-defined on ideles: for (α_v) an idele, α_v is a unit for almost all v and L_w/K_v is unramified for almost all v. For the (almost all) v in both categories, $r_{K,v}$ maps α_v to the identity.

The subtle part is the following. As noted below, before proving reciprocity, we'll only be able to check this for the map obtained from $r_{K,v}$ by projecting from $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ to $\mathrm{Gal}(K(\zeta_\infty)/K)$, the Galois group of the maximal cyclotomic extension; in that case, we can reduce to $K = \mathbb{Q}$ and do an explicit computation. The general case will actually only follow after the fact from the construction of global reciprocity!

Proposition 3. *The map $r_{K,v}$ is trivial on K^*.*

Thus it induces a map $r_K : C_K \to \mathrm{Gal}(L/K)$ for each L/K abelian, and in fact to $r_K : C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ using the analogous compatibility for local reciprocity.

Since each of the local reciprocity maps is continuous, so is the map r_K. That means the kernel of $r_K : C_K \to \mathrm{Gal}(L/K)$, for L/K abelian, is an open subgroup of C_K. Now recall that the quotient of C_K by any open subgroup of finite index is a generalized ideal class group. Thus r_K is giving us a canonical isomorphism between $\mathrm{Gal}(L/K)$ and a generalized ideal class group; could this be anything but Artin reciprocity itself?

Indeed, let U be the kernel of r_K, let $\mathfrak{m}$ be a conductor for the generalized ideal class group C_K/U, and let $\mathfrak{p}$ be a prime of K not dividing $\mathfrak{m}$ and unramified in L. Then the idele $\alpha = (1, 1, \dots, \pi, \dots)$ with a uniformizer π of $\mathfrak{o}_{K_\mathfrak{p}}$ in the $\mathfrak{p}$ component and ones elsewhere maps onto the class of $\mathfrak{p}$ in C_K/U. On the other hand, $r_K(\alpha) = r_{K,\mathfrak{p}}(\pi)$ is (because L is unramified over K) precisely the Frobenius of $\mathfrak{p}$. So indeed, $\mathfrak{p}$ is being mapped to its Frobenius, so the map $C_K/U \to \mathrm{Gal}(L/K)$ is indeed Artin reciprocity.

In fact, we discover from this a little bit more than we knew already about the Artin map. All we said before about the Artin map is that it factors through a generalized ideal class group, and that the conductor $\mathfrak{m}$ of that group is divisible precisely by the ramified primes (which we see from local reciprocity). In fact, we can now say *exactly* what is in the kernel of the classical Artin map: it is generated by

- all principal ideals congruent to 1 modulo $\mathfrak{m}$;
- norms of ideals of L not divisible by any ramified primes.

What needs to be done.

Many of these steps will be analogous to the steps in local CFT. We may have to suppress some of the proofs depending on time constraints.

- Verify that the map r_K given above does indeed kill principal ideles. That's too hard to do all at once, so we first consider the projection $r_K : K^* \to \mathrm{Gal}(K(\zeta_\infty)/K)$ onto the Galois group of the cyclotomic extension. In that case, we can reduce to $K = \mathbb{Q}$, where all the local reciprocity maps are pretty easy to compute. The fact that r_K kills principal ideles in general will have to be deduced after some sort of reciprocity law is established.

- Verify that for L/K cyclic, the Herbrand quotient of C_L as a $\mathrm{Gal}(L/K)$-module is $[L : K]$. In particular, that forces $\#H^0(\mathrm{Gal}(L/K), C_L) \geq [L : K]$ (the "First Inequality").

- For L/K cyclic, determine that

$$\#H^0(\mathrm{Gal}(L/K), C_L) = [L : K], \qquad \#H^1(\mathrm{Gal}(L/K), C_L) = 1$$

(the "Second Inequality"). This step is trivial in local CFT by Hilbert's Theorem 90 but is actually pretty subtle in the global case. We'll do it by reducing to the case where K contains enough roots of unity, so that L/K becomes a Kummer extension and we can compute everything explicitly. There is also an analytic proof given in Milne which I'll very briefly allude to.

- For a Neukirch-style completion, write down d (i.e., choose the "unramified" extensions) and the "valuation" v, and check that they satisfy the conditions for abstract class field theory.

- For a Milne-style completion, check that $H^2(\mathrm{Gal}(L/K), C_L)$ is cyclic of order $[L : K]$ in "unramified" cases. Then as in the local case, deduce the same result in general by induction on degree. Get a reciprocity map from $H_T^{-2}(\mathrm{Gal}(L/K), \mathbb{Z}) = \mathrm{Gal}(L/K)^{\mathrm{ab}}$ to $H_T^0(\mathrm{Gal}(L/K), C_K / \mathrm{Norm}_{L/K} C_L)$ using the theorem of Tate that we used before.

- Prove the existence theorem, by showing that every open subgroup of C_K contains a norm group. Again, we can enlarge K in order to do this, so we can get into the realm of Kummer theory.

Note: in either completion, the fact that the reciprocity map we get is the one we actually want follows from the fact that we are "paralleling" the local construction in the global construction. In fact, that will give us the fact that the reciprocity map that we tried to define at the outset actually does kill principal ideles!