Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
The Existence Theorem

Reference. Milne VII.9, Neukirch VI.6.

With the "abstract" reciprocity theorem in hand, we now prove the Existence Theorem, that every generalized ideal class group of a number field is identified by Artin reciprocity with the Galois group of a suitable abelian extension. The idelic formulation of this is as follows.

Theorem 1. *For K a number field, the finite abelian extensions L/K correspond one-to-one with the open subgroups of C_K of finite index, via the map $L \mapsto \mathrm{Norm}_{L/K} C_L$.*

The proof of this result will be very similar to the proof we gave in the local case. For example, the reciprocity law immediately lets us reduce to the following proposition.

Proposition 2. *Every open subgroup U of C_K of finite index contains $\mathrm{Norm}_{L/K} C_L$ for some finite extension L of K.*

Proof. We first prove this proposition in case U has prime index p. Let J be the preimage of U under the projection $I_K \to C_K$, so that J is open in I_K of finite index. Then J contains a subgroup of the form

$$V = \prod_{v \in S} \{1\} \times \prod_{v \notin S} \mathfrak{o}^*_{K_v}$$

for some set S of places of K containing the infinite places and all places dividing (p), which we may choose large enough so that $I_{K,S}K^* = I_K$. Let $K_S = K^* \cap I_{K,S}$ be the group of S-units of K.

The group J must also contain I_K^p, so in particular contains

$$W_S = \prod_{v \in S} (K_v^*)^p \times \prod_{v \notin S} U_v.$$

Put $C_S = W_S K^*/K^*$; then $C_S \subseteq U$, so it suffices to show that C_S contains a norm subgroup.

If K contains a primitive p-th root of unity, then an argument a la the algebraic proof of the Second Inequality gives $C_S = \mathrm{Norm}_{L/K} C_L$ for $L = K(K_S^{1/p})$. Namely, one first computes that $\#C_K/C_S = p^{\#S} = [L : K]$ as in that proof, by using the short exact sequence

$$1 \to \frac{K_S}{W_S \cap K_S} \to \frac{I_{K,S}}{W_S} \to \frac{C_K}{C_S} \to 1$$

and the fact that $W_S \cap K_S = K_S^p$ (a la Lemma 3 from the Second Inequality handout; remember that this uses the First Inequality), which gives $\#K_S/(W_S \cap K_S) = p^{\#S}$. Meanwhile, $I_{K,S}/W_S$ is the product of $\#S$ quotients of the form $K_v^*/(K_v^*)^p$, each of which has order p^2 (generated by a uniformizer and a p-th root of unity, since p is prime to the residue characteristic).

One then checks that $W_S \subseteq \operatorname{Norm}_{L/K} I_L$ by checking this place by place; the places not in S are straightforward (they don't ramify in L, so local units are local norms), and the ones in S follow from the fact that for any local field M containing a p-th root of unity, if $N = M((M^*)^{1/p})$, then

$$\operatorname{Norm}_{N/M} N^* = (M^*)^p,$$

which we proved in the course of proving the local existence theorem. (Reminder of proof: by local reciprocity, the group $M^*/\operatorname{Norm}_{N/M} N^* \cong \operatorname{Gal}(N/M)$ is killed by p, so it contains $(M^*)^p$; but both groups have the same index $[N : M]$.) Putting this all together, we have $C' \subseteq \operatorname{Norm}_{L/K} C_L$ and these two groups have the same index $[L : K]$ in C_K by the First and Second Inequalities, so they are equal.

We next drop the restriction that K contains a p-th root of unity by reducing to the previous case. Namely, put $K' = K(\zeta_p)$. For a choice of S as above, let S' be the set of places of K' above S; we can make S large enough so that $I_{K',S'}(K')^* = I_{K'}$. Then as above, $C_{S'} = \operatorname{Norm}_{L'/K'} C_{L'}$ if L' is the extension of K' obtained by adjoining all p-th roots. Also as above, $\operatorname{Norm}_{K'/K} W_{S'} \subseteq W_S$, so

$$\operatorname{Norm}_{L'/K} C_{L'} = \operatorname{Norm}_{K'/K}(\operatorname{Norm}_{L'/K'} C_{L'}) = \operatorname{Norm}_{K'/K} C_{S'} \subseteq C_S \subseteq U.$$

Finally, we handle the case where U has arbitrary index, by induction on that index using the above result as the base case. If $\#C_K/U$ is not prime, choose an intermediate subgroup V between U and C_K. By the induction hypothesis, V contains $N = \operatorname{Norm}_{L/K} C_L$ for some finite extension L of K. Then

$$\#N/(U \cap N) = \#UN/U \leq \#V/U.$$

Let W be the subgroup of C_L consisting of those x whose norms lie in U. Then

$$\#C_L/W \leq \#N/(U \cap N) \leq \#V/U,$$

so by the induction hypothesis, W contains $\operatorname{Norm}_{M/L} C_M$ for some finite extension M/L. Thus U contains $\operatorname{Norm}_{M/K} C_M$, as desired. $\square$