Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
Cohomology of local fields: some computations

Reference. Milne, III.2 and III.3; Neukirch, V.1.

Notation convention. If you catch me writing $H^i(L/K)$ for L/K a Galois extension of fields, that's shorthand for $H^i(\mathrm{Gal}(L/K), L^*)$. Likewise for H_i or H^i_T.

We now make some computations of $H^i_T(L/K)$ for L/K a finite Galois extension of local fields. The first one is one we already made back during Kummer theory.

Proposition 1 (Hilbert-Noether Theorem 90). *For any finite extension L/K of fields, $H^1(L/K) = 0$.*

Our ultimate goal today will be to prove the following.

Proposition 2. *For any finite Galois extension L/K of local fields, $H^2(L/K)$ is cyclic of order $[L : K]$. Moreover, this group can be canonically identified with $\frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$, such that if M/L is another finite extension such that M/K is also Galois, the inflation homomorphism $H^2(L/K) \to H^2(M/K)$ corresponds to the inclusion $\frac{1}{[M:K]}\mathbb{Z}/\mathbb{Z} \subseteq \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$.*

Before continuing, it is worth keeping in a safe place the exact sequence

$$1 \to \mathfrak{o}_L^* \to L^* \to L^*/\mathfrak{o}_L^* = \pi_L^{\mathbb{Z}} \to 1,$$

In this exact sequence of $G = \mathrm{Gal}(L/K)$-modules, the action on $\pi_L^{\mathbb{Z}}$ is always trivial (since the valuation on L is Galois-invariant). For convenience, we write U_L for the unit group $\mathfrak{o}_L^*$.

The unramified case. Recall that unramified extensions are cyclic, since their Galois groups are also the Galois groups of extensions of finite fields.

Proposition 3. *For any finite extension L/K of* finite *fields, the map $\mathrm{Norm}_{L/K} : L^* \to K^*$ is surjective.*

Proof. One can certainly give an elementary proof of this using the fact that L^* is cyclic (exercise). But one can also see it using the machinery we have at hand. Because L^* is a finite module, its Herbrand quotient is 1. Also, we know $H^1_T(L/K)$ is trivial by Hilbert Theorem 90. Thus $H^0_T(L/K)$ is trivial too, that is, $\mathrm{Norm}_{L/K} : L^* \to K^*$ is surjective. □

Proposition 4. *For any finite unramified extension L/K of local fields, the map $\mathrm{Norm}_{L/K}$: $U_L \to U_K$ is surjective.*

Proof. Say $u \in U_K$ is a unit. Pick $v_0 \in L/K$ such that in the residue fields, the norm of v_0 coincides with u. Thus $u/\mathrm{Norm}(v_0) \equiv 1 \pmod{\pi}$, where π is a uniformizer of K. Now we construct units $v_i \equiv 1 \pmod{\pi^i}$ such that $u_i = u/\mathrm{Norm}(v_0 \cdots v_i) \equiv 1 \pmod{\pi^{i+1}}$: simply take v_i so that $\mathrm{Trace}((1 - v_i)/\pi^i) \equiv (1 - u_i)/\pi^i \pmod{\pi}$. (That's possible because the trace map on residue fields is surjective, since the residue field extension is separable, or if you like by the normal basis theorem.) Then the product $v_0 v_1 \cdots$ convergest to a unit v with norm u. □

Corollary 5. *If L/K is an unramified extension of local fields, then $H^i_T(\mathrm{Gal}(L/K), U_L) = 1$ for all $i \in \mathbb{Z}$.*

Proof. For i odd, this is Theorem 90; for i even, it is the previous proposition. $\square$

Using the Herbrand quotient, we get $h(L^*) = h(U_L)h(L^*/U_L)$. The previous corollary says $h(U_L) = 1$. And

$$\begin{aligned} h(L^*/U_L) &= h(\mathbb{Z}) \\ &= \#H^0_T(\mathrm{Gal}(L/K), \mathbb{Z})/\#H^1_T(\mathrm{Gal}(L/K), \mathbb{Z}) \\ &= \#\,\mathrm{Gal}(L/K)^{\mathrm{ab}}/\#\,\mathrm{Hom}(\mathrm{Gal}(L/K), \mathbb{Z}) \\ &= [L : K]. \end{aligned}$$

Since $H^1_T(\mathrm{Gal}(L/K), L^*)$ is trivial, we conclude $H^0_T(\mathrm{Gal}(L/K), L^*)$ has order $[L : K]$. In fact, it is cyclic: the long exact sequence of Tate groups gives

$$1 = H^0_T(\mathrm{Gal}(L/K), U_K) \to H^0_T(\mathrm{Gal}(L/K), L^*) \to H^0_T(\mathrm{Gal}(L/K), \mathbb{Z}) \to H^1_T(\mathrm{Gal}(L/K), U_K) = 1$$

and the middle map is an isomorphism.

Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

of modules with trivial Galois action. Since $\mathbb{Q}$ is injective as an abelian group, it is also injective as a G-module for any group G (exercise). Thus we get an isomorphism $H^0_T(\mathrm{Gal}(L/K), \mathbb{Z}) \to H^{-1}_T(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z})$. But the latter is

$$H^1(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z});$$

since $\mathrm{Gal}(L/K)$ has a canonical generator (Frobenius), we can evaluate there and get a canonical map $\mathrm{Hom}(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \to \mathbb{Z}/[L : K]\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Putting it all together, we get a canonical map

$$H^2(\mathrm{Gal}(L/K), L^*) \cong H^0_T(\mathrm{Gal}(L/K), L^*) \cong H^1(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

In this special case, this is none other than the local invariant map! In fact, by taking direct limits, we get a canonical isomorphism

$$H^2(K^{\mathrm{unr}}/K) \cong \mathbb{Q}/\mathbb{Z}.$$

What's really going on here is that $H^0_T(\mathrm{Gal}(L/K), L^*)$ is a cyclic group generated by a uniformizer π (since every unit is a norm). Under the map $H^0_T(\mathrm{Gal}(L/K), L^*) \to \mathbb{Q}/\mathbb{Z}$, that uniformizer is being mapped to $1/[L : K]$.

The cyclic case.

Let L/K be a cyclic but possibly ramified extension of local fields. Again, $H^1_T(L/K)$ is trivial by Theorem 90, so all there is to compute is $H^0_T(L/K)$. We are going to show again that it has order n. (It's actually cyclic again, but we won't prove this just yet.)

Lemma 6. *Let L/K be a finite Galois extension of local fields. Then there is an open, Galois-stable subgroup V of $\mathfrak{o}_L$ such that $H^i(\mathrm{Gal}(L/K), V) = 0$ for all $i > 0$ (i.e., V is acyclic for cohomology).*

Proof. By the normal basis theorem, there exists $\alpha \in L$ such that $\{\alpha^g : g \in \mathrm{Gal}(L/K)\}$ is a basis for L over K. Without loss of generality, we may rescale to get $\alpha \in \mathfrak{o}_L$; then put $V = \sum \mathfrak{o}_K \alpha^g$. As in the proof of additive Theorem 90, V is induced: $V = \mathrm{Ind}^G \mathfrak{o}_K$, so is acyclic. $\square$

The following proof uses that we are in characteristic 0, but you can patch it up for the function field case.

Lemma 7. *Let L/K be a finite Galois extension of local fields. Then there is an open, Galois-stable subgroup W of $U_L = \mathfrak{o}_L^*$ such that $H^i(\mathrm{Gal}(L/K), W) = 0$ for all $i > 0$.*

Proof. Take V as in the previous lemma. If we chose α sufficiently divisible, then V lies in the radius of convergence of the exponential series

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

(you need $v_p(x) > 1/(p-1)$, to be precise), and we may take $W = \exp(V)$. $\square$

Since the quotient U_L/V is finite, its Herbrand quotient is 1, so $h(U_L) = h(V) = 1$. So again we may conclude that $h(L^*) = h(U_L)h(\mathbb{Z}) = [L : K]$, and so $H^0_T(\mathrm{Gal}(L/K), L^*) = [L : K]$. But we don't yet know that $H^0_T(\mathrm{Gal}(L/K), L^*)$ is cyclic, since the groups $H^i_T(\mathrm{Gal}(L/K), U_L)$ don't necessarily vanish.

Note. This is all that we need for "abstract" local class field theory. We'll revisit this point later.

The general case.

For those in the know, there is a spectral sequence underlying this next result; see Milne, Remark II.1.35.

Proposition 8 (Inflation-Restriction Exact Sequence). *Let G be a finite group, H a normal subgroup, and M a G-module. If $H^i(H, M) = 0$ for $i = 1, \ldots, r-1$, then*

$$0 \to H^r(G/H, M) \stackrel{\mathrm{Inf}}{\to} H^r(G, M) \stackrel{\mathrm{Res}}{\to} H^r(H, M)$$

is exact.

Proof. For $r = 1$, the condition on H^i is empty. In this case, $H^1(G, M)$ classifies crossed homomorphisms $\phi : G \to M$. If one of these factors through G/H, then clearly it becomes trivial when restricted to H. Conversely, if $\phi(h) = m^h - m$ for all $h \in H$, then $g \mapsto \phi(g) - m^g + m$ is a crossed homomorphism which factors through G/H. Thus the sequence is exact at $H^1(G, M)$; exactness at $H^i(G/H, M)$ is similar but easier.

If $r > 1$, we induct on r by dimension shifting. Let N be the G-module which makes the sequence

$$0 \to M \to \mathrm{Ind}^G(M) \to N \to 0.$$

By the long exact sequence in homology, $H^i(N) \cong H^{i+1}(M)$. Thus the case of r for M reduces to the case of $r-1$ for N. $\square$

By Theorem 90, we thus have

Corollary 9. *If $M/L/K$ is a tower of fields with M/K and L/K finite and Galois, the sequence*

$$0 \to H^2(L/K) \to H^2(M/K) \to H^2(M/L)$$

is exact.

We now prove the following.

Proposition 10. *For any finite Galois extension L/K of local fields, $H^2(\mathrm{Gal}(L/K), L^*)$ has order at most $[L : K]$.*

A key fact we need to recall is that any finite extension of local fields is *solvable*: the maximal unramified extension is cyclic, the maximal tamely ramified extension is cyclic over that, and the rest is an extension of order a power of p, so its Galois group is automatically solvable. This lets us induct on $[L : K]$.

Proof. We've checked the case L/K cyclic, so we may use it as the basis for an induction. If L/K is not cyclic, since it is solvable, we can find a Galois subextension M/K. Now the exact sequence

$$0 \to H^2(M/K) \to H^2(L/K) \to H^2(L/M)$$

implies that $\#H^2(L/K) \le \#H^2(M/K)\#H^2(L/M) = [M : K][L : M] = [L : K]$. $\square$

To complete the proof that $H^2(L/K)$ is cyclic of order $[L : K]$, it now suffices to produce a cyclic subgroup of order n. Let M/K be an unramified extension of degree n. Then we have a diagram

$$\begin{tikzcd}
 & & H^2(M/K) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{Res}"] & H^2(ML/L) \arrow[d, "\mathrm{Res}"] \\
0 \arrow[r] & H^2(L/K) \arrow[r] & H^2(ML/K) \arrow[r] & H^2(ML/L)
\end{tikzcd}$$

in which the bottom row is exact and the vertical arrows are injective, both by the inflation-restriction sequence. It suffices to show that $H^2(M/K) \to H^2(ML/L)$ is the zero map; then we can push a generator of $H^2(M/K)$ down to $H^2(ML/K)$, and pull it back to $H^2(L/K)$ by exactness to get an element of order n.

Let $e = e(L/K)$ and $f = f(L/K)$ be the ramification index, so that $[ML : L] = e$. Since unramified extensions are cyclic, we have $H^2(M/K) \cong H^0_T(M/K)$ and $H^2(ML/L) \cong$

$H^0_T(ML/L)$, so the map $\mathrm{Res} : H^2(M/K) \to H^2(ML/L)$ can be computed as $\mathrm{Res} : H^0_T(M/K) \to H^0_T(ML/L)$, which is just the canonical map $K^*/\operatorname{Norm}_{M/K}(M^*) \to L^*/\operatorname{Norm}_{ML/L}((ML)^*)$. Now $K^*/\operatorname{Norm}_{M/K}(M^*)$ is a cyclic group of order ef generated by π_K, a uniformizer of K, and $L^*/\operatorname{Norm}_{ML/L}((ML)^*)$ is a cyclic group of order e generated by π_L, a uniformizer of L. But π^K is a unit of $\mathfrak{o}_L$ times π_L^e. So the Res map is indeed zero.

Note. If L/K is a finite extension of degree n, then the map $\mathrm{Res} : H^2(K^{\mathrm{unr}}/K) \to H^2(L^{\mathrm{unr}}/L)$ translates by the local reciprocity map into a map from $\mathbb{Q}/\mathbb{Z}$ to itself. This map turns out to be multiplication by n (see Milne, Proposition II.2.7).

The local invariant map.

By staring again at the above argument, we can in fact prove that $H^2(\overline{K}/K) \cong \mathbb{Q}/\mathbb{Z}$. First of all, we have an injection $H^2(K^{\mathrm{unr}}/K) \to H^2(\overline{K}/K)$ by the inflation-restriction exact sequence, and the former is canonically isomorphic to $\mathbb{Q}/\mathbb{Z}$; so we have to prove that this injection is actually also surjective. Remember that $H^2(\overline{K}/K)$ is the direct limit of $H^2(M/K)$ running over all finite extensions M of K. What we just showed above is that if $[M : K] = n$ and L is the unramified extension of K of degree n, then the images of $H^2(M/K)$ and $H^2(L/K)$ in $H^2(ML/K)$ are the same. In particular, that means that $H^2(M/K)$ is in the image of the map $H^2(K^{\mathrm{unr}}/K) \to H^2(\overline{K}/K)$. Since that's true for any M, we get that the map is indeed surjective, hence an isomorphism.

Next time, we'll use this map to obtain the local reciprocity map.

Exercises.

1. Give an elementary proof (without cohomology) that the norm map from one finite field to another is always surjective.

2. Give an example of a cyclic ramified extension L/K of local fields in which the groups $H^i_T(\mathrm{Gal}(L/K), U_L)$ are nontrivial.