Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
Overview of local class field theory

Reference. Milne, I.1; Neukirch, V.1.

We will spend the next few lectures establishing *local class field theory*, a classification of the abelian extensions of a local field. This will serve two purposes. On one hand, the results of local class field theory can be used to assist in the proofs of the global theorems, as we saw with Kronecker-Weber. On the other hand, they also give us a model set of proofs which we will attempt to emulate in the global case.

Recall that the term "local field" refers to a finite extension either of the field of p-adic numbers $\mathbb{Q}_p$ or of the field of power series $\mathbb{F}_q((t))$. I'm going to abuse language and ignore the second case, although all but a few things I'll say go through in the second case, and I'll try to flag those when they come up. (One big one: a lot of extensions have to be assumed to be separable for things to work right.)

The local reciprocity law.

The main theorem of local class field theory is the following. For K a local field, let K^{ab} be the maximal abelian extension of K.

Theorem 1 (Local Reciprocity Law). *Let K be a local field. Then there is a unique map $\phi_K : K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ satisfying the following conditions:*

(a) for any generator π of the maximal ideal of $\mathfrak{o}_K$ and any finite unramified extension L of K, $\phi_K(\pi)$ acts on L as the Frobenius automorphism;

(b) for any finite abelian extension L of K, the group of norms $\mathrm{Norm}_{L/K}(L^*)$ is in the kernel of ϕ_K, and the induced map $K^*/\mathrm{Norm}_{L/K}(L^*) \to \mathrm{Gal}(L/K)$ is an isomorphism.

The map ϕ_K is variously called the local reciprocity map or the norm residue symbol. Using the local Kronecker-Weber theorem, this can be explicitly verified for $K = \mathbb{Q}_p$ (see exercises).

The local reciprocity law is an analogue of the Artin reprocity law for number fields. We also get an analogue of the existence of class fields.

Theorem 2 (Local existence theorem). *For every finite (not necessarily abelian) extension L of K, $\mathrm{Norm}_{L/K}(L^*)$ is an open subgroup of K of finite index. Conversely, for every (open) subgroup U of K^* of finite index, there exists a finite abelian extension L of K such that $U = \mathrm{Norm}_{L/K}(L^*)$.*

(The condition "open" is only needed in the function field case; for K a finite extension of $\mathbb{Q}_p$, one can show that every subgroup of K^* of finite index is open.) The above theorem says that if we start with a nonabelian extension L, then $\mathrm{Norm}_{L/K}(L^*)$ is also the group of norms of an abelian extension. Which one?

Theorem 3 (Norm limitation theorem). *Let M be the maximal abelian subextension of L/K. Then $\mathrm{Norm}_{L/K}(L^*) = \mathrm{Norm}_{M/K}(M^*)$.*

Aside: for each uniformizer (generator of the maximal ideal) π of K, let K_π be the composite of all finite abelian extensions L such that $\pi \in \mathrm{Norm}_{L/K}(L^*)$. Then the local reciprocity map implies that $K^{\mathrm{ab}} = K_\pi \cdot K^{\mathrm{unr}}$. It turns out that K_π can be explicitly constructed as the extension of K by certain elements, thus giving a generalization of local Kronecker-Weber to arbitrary local fields! These elements come from Lubin-Tate formal groups, which we will not discuss further (but is a possible final paper topic).

Note that for L/K a finite extension of local fields, the isomorphism $K^*/\mathrm{Norm}_{L/K}(L^*) \to \mathrm{Gal}(L/K) = G$ from the local reciprocity law is in fact an isomorphism of $G = G^{\mathrm{ab}} = H_T^{-2}(G, \mathbb{Z})$ to $K^*/\mathrm{Norm}_{L/K}(L^*) = H_T^0(G, L^*)$. We will in fact show something stronger, from which we will deduce the local reciprocity law.

Theorem 4. *For any finite Galois extension L/K of local fields with Galois group G, there is a canonical isomorphism $H_T^i(G, \mathbb{Z}) \to H_T^{i+2}(G, L^*)$.*

In fact, this map can be written in terms of the cup product in group cohomology, which we have not defined (and will not).

The local invariant map.

One way to deduce the local reciprocity law (the one we will carry out first) is to first prove the following.

Theorem 5. *For any local field K, there exist canonical isomorphisms*

$$H^2(\mathrm{Gal}(K^{\mathrm{unr}}/K), (K^{\mathrm{unr}})^*) \to H^2(\mathrm{Gal}(\overline{K}/K), \overline{K}^*)$$
$$\mathrm{inv}_K : H^2(\mathrm{Gal}(\overline{K}/K), \overline{K}^*) \to \mathbb{Q}/\mathbb{Z}.$$

The first map is an inflation homomorphism; the second map in this theorem is called the *local invariant map*. More precisely, for L/K finite of degree n, we have an isomorphism

$$\mathrm{inv}_{L/K} : H^2(\mathrm{Gal}(L/K), L^*) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

and these isomorphisms are compatible with inflation. (In particular, we don't need to prove the first isomorphism separately. But that can be done, by considerations involving the Brauer group; see below.)

To use this to prove Theorem 4 and hence the local reciprocity law, we employ the following theorem of Tate, which we will prove a bit later.

Theorem 6. *Let G be a finite group and M a G-module. Suppose that for each subgroup H of G (including $H = G$), $H^1(H, M) = 0$ and $H^2(H, M)$ is cyclic of order $\#H$. Then there exist isomorphisms $H_T^i(G, \mathbb{Z}) \to H_T^{i+2}(G, M)$ for all i; these are canonical once you fix a choice of a generator of $H^2(G, M)$.*

In general, for any field K, the group $H^2(\mathrm{Gal}(\overline{K}/K), \overline{K}^*)$ is called the *Brauer group* of K. It is an important invariant of K; it can be realized also in terms of certain noncommutative algebras over K (central simple algebras). I won't pursue this connection further, nor study many of the interesting properties and applications of Brauer groups, but this is a possible final paper topic.

Abstract class field theory.

Having derived local class field theory once, we will do it again a slightly different way. In the course of proving the above results, we will have calculated that if L/K is a cyclic extension of local fields, that

$$\#H_T^0(\mathrm{Gal}(L/K), L^*) = [L : K], \qquad \#H_T^{-1}(\mathrm{Gal}(L/K), L^*) = 1.$$

It turns out that this alone is enough number-theoretic input to prove local class field theory! More precisely, given a field K with $G = \mathrm{Gal}(\overline{K}/K)$, a continuous G-module A a surjective continuous homomorphism $d : G \to \widehat{\mathbb{Z}}$, and a homomorphism $v : A^G \to \widehat{\mathbb{Z}}$ satisfying suitable conditions, we will show that for every finite extension L of K there is a canonical isomorphism $\mathrm{Gal}(L/K)^{\mathrm{ab}} \to A_L \to \mathrm{Norm}_{L/K} A_K$, where A_K and A_L denote the $\mathrm{Gal}(\overline{K}/K)$ and $\mathrm{Gal}(\overline{K}/L)$-invariants of A. In particular, these conditions will hold for K a local field, $A = \overline{K}^*$, d the map $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(K^{\mathrm{unr}}/K)$, and $v : \mathrm{Gal}(K^*) \to \mathbb{Z} \to \widehat{\mathbb{Z}}$ the valuation.

This is the precise sense in which we will use local class field theory as a model for global class field theory. After we complete local class field theory, our next goal will be to construct an analogous module A in the global case which is "complete enough" that its H_T^0 and H_T^{-1} will not be too big; the result will be the idele class group. (One main difference is that in the global case, the analogue of v will really take values in $\widehat{\mathbb{Z}}$, not just $\mathbb{Z}$.)

Exercises.

1. For $K = \mathbb{Q}_p$, the local reciprocity map plus the local Kronecker-Weber theorem give a canonical map $\mathbb{Q}_p^* \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \cong \widehat{\mathbb{Z}}$. What is the map? Optional: turn this around and deduce local Kronecker-Weber from local reciprocity.

2. For $K = \mathbb{Q}_p$, take $\pi = p$. Determine K_π, again using local Kronecker-Weber.

3. Prove that for any finite extension L/K of finite extensions of $\mathbb{Q}_p$, $\mathrm{Norm}_{L/K}(L^*)$ is an open subgroup of K^*. (Hint: show that already $\mathrm{Norm}_{L/K}(K^*)$ is open!)