Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
Parting Thoughts

Class field theory is a vast expanse of mathematics, so it's worth concluding by taking stock of what we've seen and what we haven't. First, a reminder of the main topics we have covered:

- The Kronecker-Weber theorem: the maximal abelian extension of $\mathbb{Q}$ is generated by roots of unity.
- The Artin reciprocity law for an abelian extension of a number field.
- The existence theorem classifying abelian extensions of number fields in terms of generalized ideal class groups.
- The Chebotarev density theorem, describing the distribution over primes of a number field of various splitting behaviors in an extension field.
- Some group cohomology "nuts and bolts", including some key results of Tate.
- The local reciprocity law and existence theorem.
- Adeles, ideles, and the idelic formulations of reciprocity and the existence theorem.
- Computations of group cohomology in the local (multiplicative group) and global (idele class group) cases.

Now, some things that we haven't covered that you'll find in the final papers.

- The Lubin-Tate construction of explicit class field theory for local fields (Tathagata).
- The Brauer group of a field (i.e., $H^2(\mathrm{Gal}(\overline{K}/K), K^*)$), its relationship with central simple algebras, and the Fundamental Exact Sequence (Ulrich).
- More details about zeta functions and L-functions, including the class number formula (Jose) and the distribution of norms in ideal classes (Carl).
- Another application of group cohomology: to computing ranks of elliptic curves (Dragos).
- Orders in number fields, and the notion of a "ring class field" (Pace).
- An analogue of the Kronecker-Weber theorem over the function field $\mathbb{F}_q(t)$, and even over its extensions (Li-Chung).
- Explicit class field theory for imaginary quadratic fields, via elliptic curves with complex multiplication (Mariana).

- Quadratic forms over number fields and the Hasse-Minkowski theorem (Nirit).
- Artin (nonabelian) L-series, the basis of "nonabelian class field theory" (James).

And some more topics you won't find there either. Feel free to expand this list yourself.

- The Golod-Shafarevich inequality and the class field tower problem (see Cassels-Frohlich).
- Class field theory for function fields used to produce curves over finite fields with unusually many points (see the web site of the Arizona Winter School).
- Application of Artin reciprocity to cubic, quartic and higher reciprocity (see Milne).
- Algorithmic class field theory (see the books of Henri Cohen).

And finally, some ruminations about where number theory has gone in the fifty or so years since the results of class field theory were established in the form that we saw them. In its cleanest form, class field theory describes a correspondence between one-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$, for K a number field, and certain representations of $\mathrm{GL}_1(\mathbb{A}_K)$, otherwise known as the group of ideles. But what about the nonabelian extensions of K, or what is about the same, the higher-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$?

In fact, building on work of many authors, Langlands has proposed that for every n, there should be a correspondence between n-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ and representations of $\mathrm{GL}_n(\mathbb{A}_K)$. This correspondence is the heart of the so-called "Langlands Program", an unbelievably deep web of statements which has driven much of the mathematical establishment for the last few decades. For example, for $n = 2$, this correspondence includes on one hand the 2-dimensional Galois representations coming from elliptic curves, and on the other hand representations of $\mathrm{GL}_2(\mathbb{A}_K)$ corresponding to modular forms. In particular, it includes the "modularity of elliptic curves", proved by Breuil, Conrad, Diamond and Taylor following on the celebrated work of Wiles on Fermat's Last Theorem.

Various analogues of the Langlands correspondence have been worked out very recently: for local fields by Taylor and Harris (and again, more simply, by Henniart), and for function fields by Lafforgue, based on the work of Drinfeld.

Okay, enough rambling for now; I hope that helps provide a bit of perspective. Thanks for attending!