**Math 254B, UC Berkeley, Spring 2002 (Kedlaya)**
**Zeta Functions and the Chebotarev Density Theorem**

**Reference**. Lang, *Algebraic Number Theory*, Chapter VIII (shudder) for starters; see also Milne, Chapter VI and Neukirch, Chapter VII. For advanced reading, see Tate's thesis (last chapter of Cassels-Frohlich), but wait until we introduce the adeles.

Although this is supposed to be a course on algebraic number theory, the following analytic discussion is so fundamental that we must at least allude to it here.

Let $K$ be a number field. The *Dedekind zeta function* $\zeta_K(s)$ is a function on the complex plane given, for $\mathrm{Re}(s) > 1$, by the absolutely convergent product and sum

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - \mathrm{Norm}(\mathfrak{p})^{-s})^{-1} = \zeta_K(s) = \sum_{\mathfrak{a}} \mathrm{Norm}(\mathfrak{a})^{-s},$$

where in the sum $\mathfrak{a}$ runs over the nonzero ideals of $\mathfrak{o}_K$.

A fundamental fact about the zeta function is the following. We omit the proof.

**Theorem 1.** *The function $\zeta_K(s)$ extends to a meromorphic function on $\mathbb{C}$ whose only pole is a simple pole at $s = 1$ of residue 1.*

The case $K = \mathbb{Q}$ is of course the famous Riemann zeta function. There is also a functional equation relating the values of $\zeta_K$ at $s$ and $1-s$, and an extended Riemann hypothesis: aside from "trivial" zeros along the negative real axis, the zeroes of $\zeta_K$ all have real part $1/2$.

More generally, let $\mathfrak{m}$ be a formal product of places of $K$, and let $\chi_{\mathfrak{m}} : \mathrm{Cl}_{\mathfrak{m}}(K) \to \mathbb{C}^*$ be a character of the ray class group of conductor $\mathfrak{m}$. Extend $\chi_{\mathfrak{m}}$ to a function on all ideals of $K$ by declaring its value to be 0 on ideals not coprime to $\mathfrak{m}$. Then we define the $L$-function

$$L(s, \chi_{\mathfrak{m}}) = \prod_{\mathfrak{p} \nmid \mathfrak{m}} (1 - \chi(\mathfrak{p}) \, \mathrm{Norm}(\mathfrak{p})^{-s})^{-1} = \sum_{(\mathfrak{a}, \mathfrak{m}) = 1} \chi(\mathfrak{a}) \, \mathrm{Norm}(\mathfrak{a})^{-s}.$$

Then we have another basic fact whose proof we also omit.

**Theorem 2.** *If $\chi_{\mathfrak{m}}$ is not trivial, then $L(s, \chi_{\mathfrak{m}})$ extends to an analytic function on $\mathbb{C}$.*

If $\chi_{\mathfrak{m}}$ is trivial, then $L(s, \chi_{\mathfrak{m}})$ is just the Dedekind zeta function with the Euler factors for primes dividing $\mathfrak{m}$ removed, so it still has a pole at $s = 1$.

**Theorem 3.** *If $\chi_{\mathfrak{m}}$ is not the trivial character, then $L(1, \chi_{\mathfrak{m}}) \neq 0$.*

This is already a nontrivial, but important result over $\mathbb{Q}$. It implies Dirichlet's famous theorem that there are infinitely many primes in arithmetic progression, by implying that for any nontrivial $\chi_{\mathfrak{m}}$, $\sum_{\mathfrak{p}} \chi(\mathfrak{p}) \, \mathrm{Norm}(\mathfrak{p})^{-s}$ remains bounded as $s \to \infty$. In fact, we say that a set of primes $S$ in a number field $K$ has Dirichlet density $d$ if

$$\lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in S} \mathrm{Norm}(\mathfrak{p})^{-s}}{\log \frac{1}{s-1}} = d.$$

Then the fact implies that the Dirichlet density of the set of primes congruent to $a$ modulo $m$ (assuming $a$ is coprime to $m$) is $1/\phi(m)$.

The fact also implies that for any number field $K$ and any formal product of places $\mathfrak{m}$, there are infinitely many primes in each class of the ray class group of conductor $\mathfrak{m}$, the set of such primes having Dirichlet density $1/\#\operatorname{Cl}_{\mathfrak{m}}(K)$. (Proof: see exercises.)

Finally, we point out a result of class field theory that also applies to nonabelian extensions. Recall that if $L/K$ is any Galois extension of number fields with Galois group $G$, $\mathfrak{p}$ is a prime of $K$, and $\mathfrak{q}$ is a prime above $\mathfrak{p}$ which is unramified, then there is a well-defined Frobenius associated to $\mathfrak{q}$ (it's the element $g$ of the decomposition group $G_{\mathfrak{q}}$ such that $x^g \equiv x^{\#(\mathfrak{o}_K/\mathfrak{p})} \pmod{\mathfrak{q}}$); but as a function of $\mathfrak{p}$, this Frobenius is only well-defined up to conjugation in $G$.

**Theorem 4 (Chebotarev Density Theorem).** *Let $L/K$ be an arbitrary Galois extension of number fields, with Galois group $G$. Then for any $g \in G$, there exist infinitely many primes $\mathfrak{p}$ of $K$ such that there is a prime $\mathfrak{q}$ of $L$ above $\mathfrak{p}$ with Frobenius $g$. In fact, the Dirichlet density of such primes $\mathfrak{p}$ is the order of the conjugacy class of $G$ divided by $\#G$.*

*Proof.* This follows from everything we have said so far, plus Artin reciprocity, in case $L/K$ is abelian. In the general case, let $f$ be the order of $g$, and let $K'$ be the fixed field of $g$; then we know that the set of primes of $K'$ with Frobenius $g \in \operatorname{Gal}(L/K') \subset G$ has Dirichlet density $1/f$. The same is true if we restrict to primes of absolute degree 1 (see exercises).

Let $Z$ be the centralizer of $g$ in $G$; that is, $Z = \{z \in G : zg = gz\}$. Then for each prime of $K$ (of absolute degree 1) with Frobenius in the conjugacy class of $g$, there are $\#Z/f$ primes of $K'$ above it (also of absolute degree 1) with Frobenius $g$. (Say $\mathfrak{p}$ is such a prime and $\mathfrak{q}$ is a prime of $L$ above $\mathfrak{p}$ with Frobenius $g$. Then for $h \in G$, the Frobenius of $\mathfrak{q}^h$ is $hgh^{-1}$, so the number of primes $\mathfrak{q}$ with Frobenius $g$ is $\#Z$. But each prime of $L'$ below one of these is actually below $f$ of them.) Thus the density of primes of $K$ with Frobenius in the conjugacy class of $g$ is $(1/f)(1/(\#Z/f)) = 1/\#Z$. To conclude, note that the order of the conjugacy class of $G$ is $\#G/\#Z$. $\square$

**Exercises.**

1. Show that the Dirichlet density of the set of all primes of a number field is 1.

2. Show that in any number field, the Dirichlet density of the set of primes $\mathfrak{p}$ of absolute degree greater than 1 is zero.

3. Let $\mathfrak{m}$ be a formal product of places of the number field $K$. Using Theorems 1, 2 and 3, prove that the set of primes of $K$ lying in any specified class of the ray class group of conductor $\mathfrak{m}$ is $1/\#\operatorname{Cl}_{\mathfrak{m}}(K)$. (Hint: combine the quantities $\sum_{\mathfrak{p}} \chi(\mathfrak{p}) \operatorname{Norm}(\mathfrak{p})^{-s}$ to cancel out all but one class.)