Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 1 (due Wednesday, January 16)

Solve the following problems, and turn in the solutions to *four* of them. Advance warning: some problems in later sets will refer to earlier problems, so if you don't succeed in solving some problems, make sure to read the posted solutions!

General note: you are welcome to look things up as you try to solve the exercises, as long as you cite your sources. Typical sources: Hartshorne, Atiyah-Macdonald, Stacks Project, Google, Wikipedia, PlanetMath, MathOverflow.

1. Read Example 7.1.1 in Gathmann's notes. Then check that the analogous statement for $\mathbb{R}^4$ is false: the tangent bundle to the unit ball in $\mathbb{R}^4$ admits an everywhere nonzero section. Hint: use quaternions.

2. Let X be a topological space. Recall that the continuous functions from X to any given topological space form a sheaf of sets on X. In this exercise, we generalize this observation and show that in some sense it gives rise to all sheaves of sets on X.

   (a) Let Y be a second topological space and let $f : Y \to X$ be a map of sets. Define a presheaf $\mathcal{F}$ on X as follows: for each open set U of X, $\mathcal{F}(U)$ is the set of continuous functions $g : U \to Y$ such that the composition $f \circ g$ equals the natural inclusion $U \to X$. Prove that $\mathcal{F}$ is a sheaf.

   (b) Let $\mathcal{G}$ be any sheaf of sets on X. Let Y be the disjoint union of the stalks $\mathcal{G}_x$ for $x \in X$. Let $f : Y \to X$ be the map taking each element of $\mathcal{G}_x$ to x. Construct a topology on Y for which the sheaf $\mathcal{F}$ defined in (a) equals $\mathcal{G}$. (To make precise what "equals" should mean: your identification $\mathcal{F} \cong \mathcal{G}$ should have the property that for each $x \in X$, for each open set U of X, each $x \in U$, and each $g \in \mathcal{F}(U)$, the image of g in $\mathcal{G}_x$ corresponds to the point $g(x)$ of Y.)

   (c) Optional: if you did part (b) correctly, the map f should also be continuous. (Note that this condition was not necessary for (a).)

3. Read about *adjoint functors* somewhere. Then check that for any given topological space X, the functors "sheafification" from presheaves (of sets) on X to sheaves on X and "restriction" from sheaves on X to presheaves on X form the left and right members of an adjoint pair. Hint: see Exercise 7.8.1 of Gathmann's notes. (This is a typical example of how adjoint functors arise in algebra and algebraic geometry: the right adjoint functor is a restriction functor that forgets some structure of an object, while the left adjoint takes a less structured object and "promotes" it in a canonical way. This intuition should let you come up with other examples: the free abelian group on a set of generators, the polynomial ring on a set of generators, etc.)

4. Compute the cardinality of each fiber of the map $\operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$. This relies on a well-known number-theoretic result, which you may state without proof (but do read the proof if you haven't seen this before).

5. Describe the closed points of the topological space $\operatorname{Spec} \mathbb{R}[x]$.

6. (a) Prove that the pullback of the twisting sheaf $\mathcal{O}(1)$ under the degree-d Veronese embedding $\mathbb{P}^n \to \mathbb{P}^N$ is isomorphic to $\mathcal{O}(d)$.

   (b) Prove that the pullback of the twisting sheaf $\mathcal{O}(1)$ under the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{mn+m+n}$ is isomorphic to the external product $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$. (The *external product* means pull back $\mathcal{O}(1)$ on $\mathbb{P}^n$ along the first projection $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$, pull back $\mathcal{O}(1)$ on $\mathbb{P}^m$ along the second projection $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$, then tensor the two sheaves together.)

7. Let R be a nonzero ring. Prove that the following conditions are equivalent.

   (a) The space $\operatorname{Spec}(R)$ is disconnected: that is, it is the disjoint union of two open-closed proper subsets.

   (b) There exist nonzero elements e_1, e_2 of R with $e_1 + e_2 = 1$ which are *idempotent*, i.e., $e_1^2 = e_1, e_2^2 = e_2$.

   Hint: use the fact that R is isomorphic to the ring of global sections of the structure sheaf.