Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 3

1. The property of upper semicontinuity may be checked locally on X, so we may assume at once that $X = \mathrm{Spec}(R)$ is affine, so that $\mathcal{F} \cong \tilde{M}$ for $M = \mathcal{F}(X)$. The upper semicontinuity property states that for any $x \in X$, if $\dim_{\kappa(x)} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x = n$, then there exists an open neighborhood U of x in X such that $\dim_{\kappa(y)} \mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y \leq n$ for all $y \in U$. To check this, choose any elements $m_1, \dots, m_n \in \mathcal{F}_x$ which form a basis of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ over $\kappa(x)$. By Nakayama's lemma, $m_1, \dots, m_n$ generate $\mathcal{F}_x$ as a module over $\mathcal{O}_{X,x}$. Now choose some generators $m'_1, \dots, m'_k$ of M as an R-module. In $\mathcal{F}_x$, we can write $m'_i = \sum_j f_{ij} m_j$ for some $f_{ij} \in \mathcal{O}_{X,x}$. Now find an open neighborhood U of x in X such that the m_i, the f_{ij}, and the equality $m'_i = \sum_j f_{ij} m_j$ all lift to U. Then $m_1, \dots, m_n$ generate $\mathcal{F}(U)$, so they also generate $\mathcal{F}_y$ for all $y \in U$. Therefore $\dim_{\kappa(y)} \mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y \leq n$ for all $y \in U$, as desired.

2. By formally differentiation of polynomials, it is clear that $\Omega_{R[x_1,\dots,x_n]/R}$ is generated by $dx_1, \dots, dx_n$, or in other words that the natural map $R^n \to \Omega_{R[x_1,\dots,x_n]/R}$ taking the generators of R^n to $dx_1, \dots, dx_n$ is surjective. The hard part is to make sure that this map is also injective. Suppose that $f_1, \dots, f_n \in R$ are such that $f_1\, dx_1 + \cdots + f_n\, dx_n = 0$. The partial derivative $\frac{\partial}{\partial x_1}$ defines an R-linear derivation from R to R, which then factors in some fashion through $d : R \to \Omega_{R[x_1,\dots,x_n]/R}$. The resulting map $\Omega_{R[x_1,\dots,x_n]/R} \to R$ sends dx_1 to 1 and $dx_2, \dots, dx_n$ to 0, so we must have $f_1 = 0$. Similarly $f_2 = \cdots = f_n = 0$.

3. It suffices to check that for each nonnegative integer k, the residue is invariant whenever f has pole order at most k. In this case, we can formally write $f = f_k T^{-k} + \cdots + f_{-1}T^{-1} + \cdots$, and then the coefficient of $T^{-1}\, dT$ in the image of $f\, dT$ under the substitution $T \mapsto a_1 T + a_2 T^2 + \cdots$ depends only on $f_{-k}, \dots, f_{-1}, a_1, \dots, a_k$. In fact, it can be written as some polynomial in these quantities with coefficients in $\mathbb{Z}$ depending only on k (not on the ring R).

   So now we must check that some specific polynomial in $f_{-k}, \dots, f_{-1}, a_1, \dots, a_k$ with integer coefficients is equal to the polynomial f_{-1}. But to check that a multivariate polynomial with integer coefficients is identically 0, it suffices to check that its evaluation at any complex numbers is zero, and this follows immediately from the Cauchy integral formula from complex analysis: the coefficient of $T^{-1}\, dT$ equals $1/(2\pi i)$ times the integral of $f\, dT$ around any simple closed curve which loops counterclockwise around 0 and is small enough not to contain any other singularities of f. Making a substitution of the form $T \mapsto a_1 T + \cdots + a_k T^k$ (there is no need to include any higher coefficients!) does not affect the looping property.

4. We use property (i) to define the residue at $P = 0$. Note that by the previous exercise, this already satisfies property (ii) for any linear fractional transformation fixing $P = 0$. Therefore, we can define the residue at any other point by using property (ii) for a

single choice of L which maps 0 to P, and the definition will not depend on the choice of L.

5. It suffices to check that for any given k, the theorem holds for $\omega = f dT$ where f is a rational function with at most k poles (counted with multiplicity). But then the claim is an algebraic identity in the coefficients of the numerator and denominator of f, and the claim that the residues sum to zero is again a statement that a certain universal polynomial with integer coefficients is identically zero. So again we may reduce to the case $k = \mathbb{C}$. In that case, we may apply a linear fractional transformation to ensure that ∞ is not a pole, then use the Cauchy integral formula to compute the sum of residues as $1/(2\pi i)$ times the integral over a simple closed curve which loops counterclockwise around all of the poles. But if we now pull back along $T \mapsto 1/T$, this curve becomes a simple closed curve which loops clockwise around no poles, so the integral must be zero.

6. Let g be the genus of C. By Riemann-Roch, $h^0(C, \mathcal{O}((g+1)P)) \geq \deg((g+1)P)+1-g > 1$, so there must be a nonconstant function which has no poles other than at P. (For $g > 0$, it would have been enough to take gP instead of $(g+1)P$.)

7. Let K_C be a canonical divisor.

   (i) Since $h^0(C, K_C) = g = 2$, K_C defines a map to $\mathbb{P}^1_k$; the degree of this map is $\deg(K_C) = 2g - 2 = 2$.

   (ii) To define the map, we must find a divisor D with $h^0(C, D) = 4$ and $\deg(D) = 5$. But $5 > 2g - 2 = K(C)$, so in fact any D with degree 5 will satisfy $h^0(C, D) = 4$. Better yet, for any two points P, Q of C, $\deg(D - P - Q) = 3 > 2g - 2$, so $h^0(C, D - P - Q) = 2$. Using the criterion described on the next homework, this implies that we get an embedding.