Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 4

1. (a) To get a map $f : V \to \mathbb{P}^{d-1}_k$, it must be the case that for each closed point P of X, the elements of V do not all have positive order of vanishing at P (so that we get a valid set of homogeneous coordinates by evaluating at P). If $V = H^0(X, \mathcal{L})$, we have an exact sequence

$$0 \to H^0(X, \mathcal{L}(-P)) \to V \to H^0(X, k_P) \cong k$$

so $h^0(X, \mathcal{L}(-P)) \in \{h^0(X, \mathcal{L}), h^0(X, \mathcal{L}) - 1\}$ with the latter case occurring if and only if there is a section of $\mathcal{L}$ failing to vanish at P.

(b) Suppose that the condition described in (a) holds. Note that the condition "there exists a section of V which vanishes at P but not at Q" is equivalent to "there exists a hyperplane in $\mathbb{P}^{d-1}_k$ which passes through $f(P)$ but not $f(Q)$". This clearly happens if and only if $f(P) \neq f(Q)$.

If $V = H^0(X, \mathcal{L})$, then there is an exact sequence

$$0 \to H^0(X, \mathcal{L}(-P - Q)) \to V \to H^0(X, k_P \oplus k_Q) \cong k \oplus k,$$

so

$$h^0(X, \mathcal{L}(-P - Q)) = h^0(X, \mathcal{L}) - 2$$

if and only if $V \to H^0(X, k_P \oplus k_Q)$ is surjective. If this map is surjective, then the inverse image of $0 \oplus 1$ is a section in V which vanishes at P but not at Q. Conversely, if there exist a section which vanishes at P but not at Q and also a section which vanishes at Q but not at P, then the images of these two sections must be linearly independent in $k_P \oplus k_Q$, so the map must be surjective.

2. This time, we have an exact sequence

$$0 \to H^0(X, \mathcal{L}(-2P)) \to V \to \mathcal{O}_{X,x}/\mathfrak{m}^2_{X,x}$$

for t a uniformizer of X at P, and

$$h^0(X, \mathcal{L}(-2P)) = h^0(X, \mathcal{L}) - 2.$$

if and only if $V \to \mathcal{O}_{X,x}/\mathfrak{m}^2_{X,x}$ is surjective. Assuming (a) from the previous exercise, this is equivalent to saying that $H^0(X, \mathcal{L}(-P)) \to \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ is surjective.

We will prove that this condition holds at a given point P if and only if f is a closed immersion on some open neighborhood of P. Since we are now working locally, we may choose a basis $s_0, \dots, s_{d-1}$ of V so that $s_1, \dots, s_{d-1}$ vanish at P but s_0 does not.

Put $R = k[s_1/s_0, \dots, s_{d-1}/s_0]$ and let I be the ideal of R generated by $s_1/s_0, \dots, s_{d-1}/s_0$. Note that f is a closed immersion in a neighborhood of P if and only if the map $R \to \mathcal{O}_{X,x}$ is surjective if and only if the map $I \to \mathfrak{m}_{X,x}$ is surjective. But by Nakayama's lemma, it is equivalent to say that $I \to \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ is surjective, so we have what we need.

3. To follow the hint, start with the curve in $\mathbb{P}^2_k$ defined by $y^2z = x^3$. This meets a typical line in 3 points, so if this is indeed the image of the map f defined by some bundle $\mathcal{L}$ on $\mathbb{P}^1_k$, that line bundle must have degree 3 and hence must be $\mathcal{O}(3)$. If we let a, b be generators of $H^0(\mathbb{P}^1_k, \mathcal{O}(1))$, then $\mathcal{O}(3)$ is generated by a^3, a^2b, ab^2, b^3, and we now wish to pick out three sections x, y, z for which $y^2z = x^3$. This suggests that we take

$$x = a^2b, y = a^3, z = b^3.$$

From the first exercise, condition (a) holds since y and z do not both vanish at any point. To check (b), note that y and z distinguish all pairs of points except those for which a/b differ by a cube root of 1, but one can then use x to separate these. But the closed immersion condition fails at the point where $a = 0$, since both x and y vanish to order 2 there.

4. If $g(X) = 0$, then for any point P viewed as a divisor, $\deg(K_X - P) = -3 < 0$, so $h^0(X, K_X - P) = 0$. By Riemann-Roch, we then have $h^0(X, P) = 2$, so there is a nonconstant function with at worst a single pole at P and no other poles. There must indeed be a pole at P or else the function would be constant. This function (or if you prefer, the line bundle $\mathcal{O}(P)$) now defines a map to $\mathbb{P}^1$ of degree 1, which is necessarily an isomorphism (because this is true at the level of local rings).

5. Supposing that $g(X) \geq 2$, to show ω_X defines a map to $\mathbb{P}^{g-1}$ we first check that $h^0(X, \omega_X(-P)) = g - 1$ for any closed point P. By Riemann-Roch,

$$h^0(X, \omega_X(-P)) - h^0(X, \mathcal{O}(P)) = (2g - 3) + (1 - g) = g - 2,$$

so it is equivalent to check that $h^0(X, \mathcal{O}(P)) = 1$. If this were to fail, then there would exist a rational function on X with only one pole, but we would then have $X \cong \mathbb{P}^1_k$ as in the previous exercise.

Suppose that X fails to be a closed immersion. This means that there exist closed points P and Q (not necessarily distinct) for which $h^0(X, \omega_X(-P-Q)) \neq h^0(X, \omega_X) - 2$. Given the previous paragraph, the only other option is $h^0(X, \omega_X(-P - Q)) = h^0(X, \omega_X) - 1$. By Riemann-Roch again, $h^0(X, \mathcal{O}(P + Q)) = 2$, so there exists a rational function on X with poles at $P + Q$. This function defines a 2-to-1 map to $\mathbb{P}^1_k$.

6. Recall that X is defined by a homogeneous polynomial of degree d, which is to say a section of $\mathcal{O}(d)$ on $\mathbb{P}^2_k$. On $\mathbb{P}^2_k$, we thus have an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(d) \to i_*\mathcal{O}_X \to 0$$

which we then twist to get

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(d - 3) \to i_*\mathcal{O}_X(d - 3) \to 0.$$

Taking global sections gives an exact sequence

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}(-3)) \to H^0(\mathbb{P}^2 \to \mathcal{O}(d - 3)) \to H^0(i_*\mathcal{O}_X(d - 3)).$$

The term $H^0(\mathbb{P}^2, \mathcal{O}(-3))$ vanishes because $\mathcal{O}(-3)$ has negative degree. Assuming that the restriction map $H^0(\mathbb{P}^2, \mathcal{O}(d-3)) \to H^0(X, \mathcal{O}(d-3))$ is surjective (which we will show later by showing that $H^1(\mathbb{P}^2, \mathcal{O}(-3))$ vanishes) we then have $g(X) = h^0(\mathbb{P}^2, \mathcal{O}(d-3)) = (d-1)(d-2)/2$.

7. If k is not algebraically closed, then a divisor on X is still defined as a formal sum of closed points. However, each closed point P should now be weighted by the degree of $\kappa(P)$ as a field extension of k.