Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 4 (due Wednesday, February 6)

Solve the following problems, and turn in the solutions to *four* of them. You may (and should) use the Riemann-Roch theorem! Throughout this problem set, let k be an algebraically closed field unless otherwise specified, and let X be a smooth, projective, connected variety of dimension 1 over k (i.e., a "curve").

1. Let $\mathcal{L}$ be a line bundle over X. Let V be a finite-dimensional subspace of $H^0(X, \mathcal{L})$ of dimension d.

   (a) I said in class that V always defines a map $f : V \to \mathbb{P}^{d-1}_k$. Actually this is not quite correct! Prove that this happens if and only if for any closed point P of X, there exists a section of V which does not vanish at P. Then check that if $V = H^0(X, \mathcal{L})$, this happens if and only if

   $$h^0(X, \mathcal{L}(-P)) = h^0(X, \mathcal{L}) - 1.$$

   Hint: the zero tuple is not a valid set of homogeneous coordinates.

   (b) Suppose that the condition described in (a) holds. Prove that f is injective on points if and only if for any two distinct closed points P, Q, there exists a section of V which vanishes at P but not at Q. Then check that if $V = H^0(X, \mathcal{L})$, this happens if and only if

   $$h^0(X, \mathcal{L}(-P - Q)) = h^0(X, \mathcal{L}) - 2.$$

2. With notation as in the previous exercise, suppose that the conditions described in (a) and (b) both hold. Then prove that f defines a closed immersion if and only if for any closed point P, there exists a section of V which vanishes at P with order exactly 1. Then check that if $V = H^0(X, \mathcal{L})$, this happens if and only if

   $$h^0(X, \mathcal{L}(-2P)) = h^0(X, \mathcal{L}) - 2.$$

3. With notation as above, construct an example where f is defined and injective on points but not a closed immersion. Hint: one possibility is to construct a map from $\mathbb{P}^1$ to $\mathbb{P}^2$ whose image is the cuspidal cubic curve $y^2 = x^3$.

4. Prove that if $g(X) = 0$, then $X \cong \mathbb{P}^1_k$. Hint: find a rational function on X with a single zero and a single pole.

5. Suppose that $g(X) \geq 2$. Using the previous exercises, prove that the canonical sheaf ω_X always defines a map $X \to \mathbb{P}^{g-1}_X$, and that this map is a closed immersion unless X admits a 2-to-1 map to $\mathbb{P}^1$ (i.e., unless X is *hyperelliptic*).

6. Let X be a smooth curve of degree d in $\mathbb{P}^2$ over k. Using the fact that $\omega_X \cong \mathcal{O}(d-3)$, prove that the genus of X equals $\frac{(d-1)(d-2)}{2}$. You may use without proof the fact that the restriction map $H^0(\mathbb{P}^2, \mathcal{O}(d-3)) \to H^0(X, \mathcal{O}(d-3))$ is surjective; we'll check this later once we have defined sheaf cohomology.

7. Suppose k is not algebraically closed. What is the correct way to define the degree of a divisor on X so that the degree of any principal divisor is still 0?