Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 7

1. If $f : X \to Y$ is finite and $Y = \mathrm{Spec}(A)$ is affine, then $f_*\mathcal{O}_X$ is a $\mathcal{O}_Y$-algebra whose underlying $\mathcal{O}_Y$-module is quasicoherent and finitely generated. By a previous homework problem, it has the form $\tilde{B}$ for some finitely generated A-module B, which also inherits a ring structure. Since $\mathcal{O}_X(X) = B$, we have a map $X \to \mathrm{Spec}(B)$; by working locally over Y we may check that this map is an isomorphism.

2. (a) In ω_X, we have the relation

$$2y\,dy = P'(x)\,dx.$$

In particular, $s = dx/(2y)$ is a section of ω_Y over the complement of $P(x) = 0$ in X and over the complement of $P'(x) = 0$ in X. But since P has no repeated roots, $P(x)$ and $P'(x)$ are coprime, and we get a section of ω_Y over all of X. For the same reason, $x^i s$ is a section of ω_Y over all of X.

It remains to check the order of vanishing at the point at infinity. For this, we note that y has $2g+1$ simple zeroes on X and so must have a pole of order $2g+1$ at ∞, while $x-c$ has 2 simple zeroes on X for most c (with some exceptions where roots come together) and so have a pole of order 2 at ∞. This means that dx has a pole of order 3 at ∞, so $x^i s$ has order of vanishing $-2i - 3 + (2g+1) = 2(g-1-i)$. We get a section of ω_Y over all of Y if and only if this order of vanishing is nonnegative, i.e., for $i = 0, \dots, g-1$.

(b) We can write any element of $H^0(X, \omega_X)$ as $(Q(x) + 2R(x)y)s$. In the cokernel of d, the term $2R(x)ys$ vanishes because it is $R(x)\,dx$, so we need only consider $Q(x)s$. But note that

$$d(x^n y) = nx^{n-1}y\,dx + x^n\,dy = (2nx^{n-1}P(x)S'(x) + x^n P'(x))s$$

and that $2nx^{n-1}P(x)S'(x) + x^n P'(x)$ is of degree exactly $n + 2g$ because the coefficient of x^{n+2g} is $2n + (2g+1) \neq 0$. So in the cokernel of d, we can rewrite $Q(x)s$ in terms of a lower-degree polynomial unless $\deg(Q) \leq 2g - 1$.

3. We start with the given exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0$$

which we already know about. We may identify the middle term with $\wedge^j \mathcal{O}(-1)^{\oplus(n+1)}$. To get the sequence

$$0 \to \Omega^j \to \mathcal{O}(-j)^{\oplus\binom{n+1}{j}} \to \Omega^{j-1} \to 0,$$

it suffices to show that for any ring R and any short exact sequence

$$0 \to M \to N \to R \to 0$$

of finite free modules, for each $j > 0$ there is a natural (i.e., not dependent on choices of bases) exact sequence

$$0 \to \wedge^j M \to \wedge^j N \to \wedge^{j-1} M \to 0.$$

The map from $\wedge^j M$ to $\wedge^j N$ is clear. To make a map from $\wedge^j N$ to $\wedge^{j-1} M$, it suffices to make a multilinear map $f : N^{\oplus j} \to \wedge^{j-1} M$ which is alternating (i.e., f evaluates to 0 whenever two of its inputs coincide). Let $\pi : N \to R$ be the map in the exact sequence. We then set

$$f(n_1, \dots, n_j) = \bigwedge_{i=2}^{j} (\pi(n_1)n_i - \pi(n_i)n_1).$$

Note that $\pi(\pi(n_1)n_i - \pi(n_i)n_1) = \pi(n_1)\pi(n_i) - \pi(n_i)\pi(n_1) = 0$, so $\pi(n_1)n_i - \pi(n_i)n_1$ may be viewed as an element of M; we thus have a map $f : N^{\oplus j} \to \wedge^{j-1} M$. Moreover, it is obvious that $f(n_1, \dots, n_j) = 0$ whenever $n_1 = n_i$ for some $i > 1$ (because then there is a zero term in the wedge product) or whenever $n_i = n_{i'}$ for some $1 < i < i' \leq j$ (because then there are two identical terms in the wedge product).

4. We will show that for $i, j = 0, \dots, n$,

$$H^i(X, \Omega^j) = \begin{cases} R & i = j \\ 0 & i \neq j. \end{cases}$$

For $j = 0$, this is the calculation of $H^i(X, \mathcal{O}_X)$ from class. For $0 < j \leq n$, using the exact sequence

$$0 \to \Omega^j \to \mathcal{O}(-j)^{\oplus \binom{n+1}{j}} \to \Omega^{j-1} \to 0,$$

from the previous exercise, we obtain a long exact sequence

$$H^{i-1}(X, \mathcal{O}(-j))^{\oplus \binom{n+1}{j}} \to H^{i-1}(X, \Omega^{j-1}) \to H^i(X, \Omega^j) \to H^i(X, \mathcal{O}(-j))^{\oplus \binom{n+1}{j}}$$

which also works for $i = 0$ if we interpret H^{-1} to be zero. Since $0 < j \leq n$, $H^i(X, \mathcal{O}(-j)) = 0$ for all i, so we have an isomorphism

$$H^{i-1}(X, \Omega^{j-1}) \cong H^i(X, \Omega^j).$$

This yields the claim by induction on j.

5. (a) By a previous homework, every sheaf injects into an injective sheaf. So we may first construct

$$0 \to \mathcal{F} \to \mathcal{I}_0.$$

By the same argument, we may construct

$$0 \to \mathcal{I}_0/\mathcal{F} \xrightarrow{d_0} \mathcal{I}_1$$

and then

$$0 \to \mathcal{I}_1/\operatorname{im}(d_0) \to \mathcal{I}_2$$

and so on.

(b) Because $\mathcal{I}_0$ is injective and $\mathcal{G} \to \mathcal{J}_0$ is an injection, we may extend $\mathcal{G} \xrightarrow{f} \mathcal{F} \to \mathcal{I}_0$ to get $f_0 : \mathcal{J}_0 \to \mathcal{I}_0$. Similarly, we may extend $J_0/\mathcal{G} \xrightarrow{f_0} \mathcal{I}_0/\mathcal{F} \xrightarrow{d_0} \mathcal{I}_1$ to a map $f_1 : \mathcal{J}_1 \to \mathcal{I}_1$ and so on.

(c) Since $f = 0$, f_0 factors through a map $\mathcal{J}_0/\mathcal{G} \to \mathcal{I}_0$. Since $\mathcal{J}_0/\mathcal{G}$ injects into $\mathcal{J}_1$, we may use the injectivity of $\mathcal{I}_0$ to extend $\mathcal{J}_0/\mathcal{G} \to \mathcal{I}_0$ to $h_1 : \mathcal{J}_1 \to \mathcal{I}_0$. Similarly, $f_1 - d_0 \circ h_1$ factors through a map $\mathcal{J}_1/\operatorname{im}(e_0) \to \mathcal{I}_1$, so we may use injectivity of $\mathcal{I}_1$ to obtain h_2 and so on.

6. (a) For a fixed choice of resolutions of $\mathcal{F}$ and $\mathcal{G}$, a map $H^i(X, \mathcal{G}) \to H^i(X, \mathcal{F})$ corresponding to f can be defined using part (b) of the previous problem. To show that this map doesn't depend on any choices made in (b), it is enough to check that if $f = 0$ then the induced maps $H^i(X, \mathcal{G}) \to H^i(X, \mathcal{F})$ are all zero. But in that case, we may use part (c) of the previous problem to write $f_i = h_{i+1} \circ e_i + d_{i-1} \circ h_i$ and then note that $h_{i+1} \circ e_i$ is zero on $\ker(\mathcal{J}_i(X) \to \mathcal{J}_{i+1}(X))$ while $d_{i-1} \circ h_i$ has image contained in $\operatorname{im}(\mathcal{I}_{i-1}(X) \to \mathcal{I}_i(X))$. So f_i is the sum of two maps which both induce the zero map $H^i(X, \mathcal{G}) \to H^i(X, \mathcal{F})$.

(b) Write $H^i(X, \mathcal{F})$ and $H^i(X, \mathcal{F})'$ for the groups computed using two different resolutions. From (a), we get maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})'$ and $H^i(X, \mathcal{F})' \to H^i(X, \mathcal{F})$. When we compose these one way, we get maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$ corresponding to the identity map $\mathcal{F} \to \mathcal{F}$. But by (a), no matter how these maps were constructed they must agree with the maps induced by taking the trivial diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}_0 & \xrightarrow{e_0} & \mathcal{I}_1 & \xrightarrow{e_1} & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}_0 & \xrightarrow{d_0} & \mathcal{I}_1 & \xrightarrow{d_1} & \cdots
\end{array}$$

in which all of the f_i are identity maps. Hence the induced map $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$ is the identity, and likewise for the composition in the other direction. We may thus view our maps as defining a distinguished isomorphism $H^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})'$.

(c) If $\mathcal{F}$ is injective, then it by itself forms an injective resolution of itself, so by definition $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

7. (a) Choose a well-ordering of I. Given a 1-cocycle $(s_{i_0 i_1})_{i_0, i_1 \in I}$ (so that $s_{i_1 i_2} - s_{i_0 i_2} + s_{i_0 i_1} = 0$ for all $i_0, i_1, i_2 \in I$), we will build a sequence of sections $u_i \in \mathcal{F}(U_i)$ for which $u_j - u_k = s_{jk}$ for all $j, k \leq i$ as follows. Suppose that the u_j have been constructed for all $j < i$. Because we started with a cocycle, the sections $u_j - s_{ji} \in \mathcal{F}(U_{ij})$ glue to an element of $\mathcal{F}(\cup_{j<i} U_{ij})$. Since $\mathcal{F}$ is flasque, this element extends to an element u_i of $\mathcal{F}(U)$ doing what we want.

(b) Construct an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{G}$ injective. By a previous exercise,

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

is exact. Since we also have an exact sequence

$$H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$$

the map $H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$ is injective. But $H^1(X, \mathcal{G}) = 0$ by the previous exercise, so $H^1(X, \mathcal{F}) = 0$.

This gives the base case of an induction on i. Given the vanishing for i, the long exact sequence gives us

$$0 = H^i(X, \mathcal{H}) \to H^{i+1}(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{G}) = 0$$

and so $H^{i+1}(X, \mathcal{F}) = 0$.