Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 9 (due *Friday*, March 15)

Solve the following problems, and turn in the solutions to *four* of them.

1. (a) Give an example of a scheme in which the intersection of two open affines fails to be affine.

   (b) Let X be a scheme in which any two points of X are contained in some open separated subscheme. Prove that X is separated.

   (c) Use (b) to give another proof that $\mathbb{P}^n_{\mathbb{Z}}$ is separated for all $n > 0$.

2. Let R be a commutative ring and let M be an R-module. A *projective resolution* of M is an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

of R-modules in which the P_i are projective. We say M has *projective dimension* m if m is the smallest nonnegative integer for which there exists a projective resolution with $P_i = 0$ for all $i > m$.

   (a) Let m be a nonnegative integer. Prove that the following conditions are equivalent.

      (i) The R-module M has projective dimension $\leq m$.

      (ii) We have $\mathrm{Ext}^{m+1}_R(M, N) = 0$ for all R-modules M.

      (iii) We have $\mathrm{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i \geq m+1$.

      (iv) For any projective resolution of M as above,

$$0 \to \mathrm{coker}(P_{m+1} \to P_m) \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$

      is also a projective resolution of M.

   Here by $\mathrm{Ext}^i_R(M, N)$, I mean the right derived functors of the contravariant functor $\mathrm{Hom}_R(\bullet, N)$. (They happens to coincide with the right derived functors of the covariant functor $\mathrm{Hom}_R(M, \bullet)$ but you shouldn't need to use this.)

   (b) Put $R = k[x]/(x^2)$ and $M = R/xR \cong k$. Prove that M does not have finite projective dimension. In particular, finite projective dimension is not guaranteed even if R is noetherian and M is finitely generated.

3. Let k be a field.

   (a) Write down the statement of the *Hilbert syzygy theorem* from Wikipedia. You do not need to include a proof.

   (b) Let $\mathcal{F}$ be a quasicoherent finitely generated sheaf on $\mathbb{P}^n_k$. Prove that for some m, there exists an exact sequence

$$0 \to \mathcal{G}_m \to \cdots \to \mathcal{G}_0 \to \mathcal{F} \to 0$$

   in which $\mathcal{G}_0, \ldots, \mathcal{G}_m$ are vector bundles.

4. Let R be the subring of $\mathbb{C}[[x_1, \dots, x_n]]$ consisting of those series which converge absolutely on some neighborhood of $(0, \dots, 0)$. By the Weierstrass preparation theorem, R is noetherian (you need not prove this). Prove that R is faithfully flat over $\mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$. Hint: compare both rings to $\mathbb{C}[[x_1, \dots, x_n]]$.

5. In this exercise, we prove part of Serre's famous GAGA theorem. Let $\mathcal{F}$ be a quasicoherent finitely generated sheaf on $\mathbb{P}^n_{\mathbb{C}}$. Let X be the analytification of $\mathbb{P}^n_{\mathbb{C}}$, and let $j : X \to \mathbb{P}^n_{\mathbb{C}}$ be the natural morphism of locally ringed spaces, as defined on the previous problem set. Assume the following facts from complex analysis.

   (i) We have $H^0(X, \mathcal{O}) = \mathbb{C}$ (by Liouville's theorem).

   (ii) We have $H^i(X, \mathcal{O}) = 0$ for all $i > 0$ (by Cartan's Theorem B).

   Prove that the maps
   $$H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}) \to H^i(X, j^*\mathcal{F}) \tag{1}$$
   derived from the universal property of derived functors are isomorphisms for all $i \geq 0$ by induction on n. Specifically, given the result for $n-1$, prove it for n in the following steps.

   (a) For any given $\mathcal{F}$, prove that the isomorphism for $\mathcal{F}$ implies the isomorphism for $\mathcal{F}(-d)$ for all $d \geq 0$, and also for all $d \geq 0$ in in case $\mathcal{F} = \mathcal{O}$.

   (b) Prove the isomorphism for $\mathcal{F}$ locally free. Hint: first prove it for $\mathcal{F}(d)$ for d large.

   (c) Prove the isomorphism for general $\mathcal{F}$.

6. In this exercise, we prove another part of Serre's GAGA theorem. Assume the following additional fact from complex analysis.[1]

   (iii) Let $\mathcal{F}$ be a sheaf on X which is locally the cokernel of an $\mathcal{O}_X$-linear map between finite free $\mathcal{O}_X$-modules. Then $H^i(X, \mathcal{F})$ is finite dimensional over $\mathbb{C}$ (by Cartan's Theorem B).

   Now take $\mathcal{F}$ as in (iii) and prove the following.

   (a) Prove that for any $x \in X$, for d sufficiently large $\mathcal{F}(d)_x$ is generated by finitely many global sections of $\mathcal{F}(d)$. Hint: draw a hyperplane through x and induct on n.

   (b) Prove that for d sufficiently large, $\mathcal{F}(d)$ is generated by finitely many global sections. Hint: the space X is compact.

   (c) Prove that $\mathcal{F} \cong j^*\mathcal{G}$ for some quasicoherent finitely generated sheaf $\mathcal{G}$ on $\mathbb{P}^n_{\mathbb{C}}$. Hint: use (b) to write $\mathcal{F}$ as a cokernel between two objects pulled back from $\mathbb{P}^n_{\mathbb{C}}$, then use the previous problem to show that the map also comes from $\mathbb{P}^n_{\mathbb{C}}$.

7. Using the GAGA exercises, prove that any analytic hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ is algebraic.

[1] This assumption was missing from the original problem statement.