Math 203B: Algebraic Geometry
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Closed subschemes

We have already defined an *open immersion* to be a morphism $f : Y \to X$ which induces an isomorphism of Y with an open subset of X. This was easy because an open subset of X inherits a scheme structure directly from X.

But from the context of varieties, we know we would also like to define *closed subschemes* of a given scheme X. This is harder because it is not obvious how to put a scheme structure on a closed set; just taking $f^{-1}\mathcal{O}_Y$ doesn't work because we don't get a locally ringed space. Also, it doesn't match what we want for varieties: we would like for instance to start with the affine plane $\operatorname{Spec} K[x, y]$, take the locus where $x = 0$, and get the affine line $\operatorname{Spec} K[y]$.

It turns out there is a good reason why this is subtle: in the category of schemes, there are usually many different "closed subspaces" with the same underlying set! For instance, in the example of the affine plane, we can also form $\operatorname{Spec} K[x, y]/(x^n)$ for any positive integer n, and this has the same underlying set as $\operatorname{Spec} K[y]$ but is not isomorphic as a scheme.

In fact, we would like to say that a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ corresponds to a closed subspace whenever $A \to B$ is a surjective morphism of rings.

Lemma 1. *Let $f : Y \to X$ be a morphism of schemes. Then the property "$Y \times_X \operatorname{Spec} A = \operatorname{Spec} B$ for some B such that $A \to B$ is surjective" is a local property of open affine subschemes $\operatorname{Spec} A$ of X.*

Proof. It is obvious that this property passes from $\operatorname{Spec} A$ to $\operatorname{Spec} A_f$. Thus we need only check that if $X = \operatorname{Spec} A$, $f_1, \dots, f_n \in A$ generate the unit ideal, and $Y \times_X \operatorname{Spec} A_{f_i} = \operatorname{Spec} B_i$ for some ring B_i such that $A_{f_i} \to B_i$ is surjective, then $Y = \operatorname{Spec} B$ for some ring B such that $A \to B$ is surjective.

There are various ways to see this, but one elegant way uses what we know about quasicoherent sheaves. Note that the kernel of a map $\mathcal{F} \to \mathcal{G}$ of quasicoherent sheaves is again quasicoherent: it locally corresponds to the kernel at the level of modules. (Warning: this is again true for cokernels, but it is not obvious because taking quotients of sheaves involves a sheafification step. We'll discuss this again shortly.)

Let $\mathcal{I}$ be the sheaf $\ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$; by the previous discussion, it is quasicoherent, and hence corresponds to an A-module I via the third fundamental theorem of schemes. Again, since kernels between modules and quasicoherent sheaves correspond, the map $I \to A$ is an inclusion, so I may be viewed as an ideal of A. Put $B = A/I$; from the isomorphisms $Y \times_X \operatorname{Spec} A_{f_i} = \operatorname{Spec} B_i \cong \operatorname{Spec} B_{f_i}$, we may assemble an isomorphism $Y \cong \operatorname{Spec} B$. □

We therefore define a *closed immersion* to be any morphism $f : Y \to X$ of schemes such that for some (hence any) open covering of X by affine schemes $\operatorname{Spec} A$, for each A we have $Y \times_X \operatorname{Spec} A = \operatorname{Spec} B$ for some ring B for which $A \to B$ is surjective. (The definition in Hartshorne is slightly different and ultimately equivalent; we will reconcile them a bit later.)

Let us again emphasize the fact that while the image of a closed immersion is indeed a closed subset of X, it is not determined by that image. For example, consider the diagram

$$\begin{array}{ccccccc} \operatorname{Spec} K[x,y]/(x) & \longrightarrow & \operatorname{Spec} K[x,y]/(x^2) & \longrightarrow & \operatorname{Spec} K[x,y]/(x^3) & \longrightarrow & \cdots \\ & \searrow & \downarrow & \swarrow & & & \\ & & \operatorname{Spec} K[x,y] & & & & \end{array}$$

in which all of the arrows are closed immersions. The first object in the top row corresponds to the "reduced" y-axis, whereas the later objects correspond to various "infinitesimally thicker" copies of the y-axis.