Math 203B: Algebraic Geometry
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Differentials and smoothness

1 Differentials

Recall that for $A \to B$ a morphism of rings, the *module of relative (Kähler) differentials* $\Omega_{B/A}$ is defined as a solution of the following universal problem. Let $D : B \to M$ be an *A-linear derivation*, i.e., a map satisfying the conditions:

- $D(b_1 + b_2) = D(b_1) + D(b_2)$ for all $b_1, b_2 \in B$;
- $D(b_1 b_2) = D(b_1)b_2 + b_1 D(b_2)$ for all $b_1, b_2 \in B$;
- $D(a) = 0$ for all $a \in A$.

Then $\Omega_{B/A}$ must comes equipped with an A-linear derivation $d : B \to \Omega_{B/A}$ such that any D as above factors uniquely through a B-linear map $\Omega_{B/A} \to M$.

For example, if $B = A[x_1, \dots, x_n]$, then we may take $\Omega_{B/A}$ to be the free module on $dx_1, \dots, dx_n$ with d given by the formal chain rule:

$$dP = \frac{\partial P}{\partial x_1} dx_1 + \cdots + \frac{\partial P}{\partial x_n} dx_n,$$

and it is easy to verify the universal property. (Namely, it is clear that dx_i must map to $D(x_i)$; since $\Omega_{B/A}$ is free, that condition defines a unique B-linear map, and that map does in fact work.)

In fact, we can always build $\Omega_{B/A}$ concretely by taking the quotient of the free module on symbols db by the relations needed to force $d : B \to \Omega_{B/A}$ to be a derivation. More elegantly, we can take it to be I/I^2 where I is the kernel of the multiplication map $B \otimes_A B \to B$, with $d(b)$ being the image of $b \otimes 1 - 1 \otimes b$.

This construction immediately extends to schemes: there is a unique way (up to unique isomorphism) to associate to each morphism $f : Y \to X$ of schemes a quasicoherent sheaf $\Omega_{Y/X}$ in such a way that the construction is functorial with respect to base change, and in the case $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ gives the sheaf associated to $\Omega_{B/A}$.

Theorem 1. *Take $X = \operatorname{Spec} R$ and $Y = \mathbb{P}^m_R$. Then $\Omega_{Y/X}$ is coherent and locally free of rank m, and there is an exact sequence*

$$0 \to \Omega_{Y/X} \to \mathcal{O}_Y(-1)^{\oplus m+1} \to \mathcal{O}_Y \to 0.$$

Proof. The first claim is immediate from our previous calculation involving the polynomial ring. To prove the second claim, we will instead produce the exact sequence

$$0 \to \Omega_{Y/X}(1) \to \mathcal{O}_Y^{\oplus m+1} \to \mathcal{O}_Y(1) \to 0.$$

The sheaf in the middle is free on $m+1$ generators which we call $dx_0, \ldots, dx_n$. Then we can define a map

$$\Omega_{Y/X}(1)(D_+(x_i)) \to \mathcal{O}_Y^{\oplus m+1}(D_+(x_i))$$

that takes $x_i d(x_j/x_i)$ to $dx_j - (x_j/x_i)dx_i$. We then define the map

$$\mathcal{O}_Y^{\oplus m+1}(D_+(x_i)) \to \mathcal{O}_Y(1)(D_+(x_i))$$

taking dx_j to x_j. One checks that this gives an exact sequence

$$0 \to \Omega_{Y/X}(1)(D_+(x_i)) \to \mathcal{O}_Y^{\oplus m+1}(D_+(x_i)) \to \mathcal{O}_Y(1)(D_+(x_i)) \to 0$$

and that the maps agree on overlaps, so they give a well-defined exact sequence of sheaves. $\square$

Note that if B is a finitely generated A-algebra, then $\Omega_{B/A}$ is a finitely generated B-module (generated by dT as t runs over some algebra generators of B/A). This also globalizes: if $f : Y \to X$ is *locally of finite type* (see homework), then $\Omega_{Y/X}$ is coherent.

Let X be a variety of dimension d over an algebraically closed field K. (If you allow reducible varieties, then assume every irreducible component has the *same* dimension d, i.e., X is of *pure dimension* d.) We say X/K is *smooth* if $\Omega_{X/K}$ is locally free of rank d. Note that this is equivalent to the probably more familiar Jacobian criterion for smoothness: locally, X embeds into an affine space $\mathbb{A}_K^m$ in such as way as to be cut out by $m - d$ polynomials whose gradients are linearly independent.