Math 203B: Algebraic Geometry
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Curves of low genus

With the Riemann-Roch theorem in hand, we study some constructions leading to curves of particular genera. Again, let k be an algebraically closed field.

1 Hyperelliptic curves

A *hyperelliptic curve* is a curve C admitting a finite morphism $f : C \to \mathbb{P}^1_k$ of degree 2. For example, every affine curve of the form $y^2 = P(x)$ in $\mathbb{A}^2_k$ extends to a hyperelliptic curve, with x defining the map to $\mathbb{P}^1_K$. It will follow from the Riemann-Hurwitz formula (see below) that if $\deg P = d$, then the genus of C equals $\lceil \frac{d}{2} - 1 \rceil$; in particular, this proves that the genus of a curve can take any nonnegative integer value.

For example, if $\deg P = 1$ one obviously gets $\mathbb{P}^1_k$ by eliminating x; if $\deg P = 2$ one has a conic section; if $\deg P = 3$ one gets a smooth cubic curve in $\mathbb{P}^2_k$. For $\deg P > 3$, this affine curve does not extend smoothly in $\mathbb{P}^2_k$, so the genus formula for smooth plane curves does not apply!

2 Riemann-Hurwitz formula

Theorem 1. *Let $f : C_1 \to C_2$ be a finite morphism of degree n. (In positive characteristic, we have to assume that f is separable, i.e., that $k(C_1)/k(C_2)$ is not only finite but also separable as a field extension.) Then*

$$2g(C_1) - 2 = n(2g(C_2) - 2) + \deg R$$

where R is a divisor associated to $(f^*\Omega_{C_2/k})^\vee \otimes \Omega_{C_1/k}$.

More precisely, we have an exact sequence

$$0 \to f^*\Omega_{C_2/k} \to \Omega_{C_1/k} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a sheaf supported at finitely many points; we may canonically (i.e., not just up to equivalence) take R to be the *ramification divisor*, i.e., the divisor consisting of the points of the support of $\mathcal{F}$, each point P occurring with multiplicity equal to the length of $\mathcal{F}_P$ as a module over $\mathcal{O}_{C,P}$. (Note: the formula now proves itself!)

For example, if $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is the map $x \mapsto z = x^2$, $P \in C_1$ is the point $x = 0$, and $Q \in C_1$ is the point $z = 0$, then $\Omega_{C_1/k,Q}$ is generated by dz, which pulls back to $d(x^2) = 2xdx$. If k is not of characteristic 2, then this means that R contains P with multiplicity 1; similarly, the point $P' \in C_1$ where $x = \infty$ is also contained in R with multiplicity 1. With this, the arithmetic works out:

$$2g(C_1) - 2 = -2 = 2(-2) + 2 = n(2g(C_2) - 2) + \deg R.$$

3 Characteristic zero versus characteristic p

In characteristic zero, it is very easy to compute the divisor R. Namely, if $P \in C_1$ mapsto $Q \in C_2$, a uniformizer $t_Q \in \mathcal{O}_{C_2,Q}$ pulls back to an element of the form $t_P^m u$ for $t_P \in \mathcal{O}_{C_1,P}$ a uniformizer, m a positive integer, and $u \in \mathcal{O}_{C_1,P}$ a unit. We then have

$$f^*(dt_Q) = \left(m t_P^{m-1} u dt_P + t_p^m \frac{du}{dt_P} \right) dt_P.$$

Since $m \neq 0$ in k, R has multiplicity $m-1$ at P.

Another way to interpret this is that R consists of the "missing preimages": most points of C_2 have exactly n distinct preimages in C_1, but a few fall short, and

$$\deg(R) = \sum_{Q \in C_2} (n - \# f^{-1}(Q)).$$

This can also be used to give a topological proof of Riemann-Hurwitz over $\mathbb{C}$: If U is the complement in C_2 of the image of the support of R, then $f^{-1}(U) \to U$ is everywhere n-to-1, so we have an equality of topological Euler characteristics:

$$\chi(f^{-1}(U)) = n\chi(U).$$

Since Euler characteristics are additive over writing a topological space as a union of an open subspace and its complement, and a point has Euler characteristic 1, this yields the proof. (Another way to interpret this is as a proof that the genus in Riemann-Roch coincides with the topological genus: we know this for $\mathbb{P}^1_{\mathbb{C}}$, and this derivation implies that both genera transform the same way under finite morphisms.)

This still works in characteristic p if none of the integers m is divisible by p; in this case we say f is *tamely ramified* (e.g., the squaring map example when $p \neq 2$). If this fails (and f is separable), we say f is *wildly ramified*; these often arise from Artin-Schreier field extensions (see homework).

4 Linear systems

If $\mathcal{L}$ is a line bundle on C and V is a subspace of $H^0(C, \mathcal{L})$ of dimension n, we've seen in a previous homework that we can attempt to define a map $C \to \mathbb{P}^{n-1}_k$ using the sections of V; this works provided that the divisors of the nonzero elements of V have no common point. (Classical terminology: the projectivization of V, or the corresponding collections of divisors, is called a *linear system* on C. A common point in the divisors is called a *base point* or *basepoint.* If there are no base points, we say V is *basepoint-free.*)

So let's try this using the canonical sheaf Ω, taking V to be the whole space of sections (which has dimension g). If $g = 0$, then $V = 0$ and this completely fails. If $g = 1$, then V is a one-dimensional space; it is basepoint-free since any section has degree $2g - 2 = 0$, but we just get a map to a point.

This gets more interesting once g gets up to 2. In this case, V is a two-dimensional space, so we potentially are getting a map $C \to \mathbb{P}^1_k$, at least provided that there is no basepoint. (In fact, the canonical linear system is always basepoint-free for $g \geq 2$; see homework.) The degree of this map can be interpreted as the degree of any nonzero divisor in the linear system, which in this case is $2g - 2 = 2$. So in fact, the one construction we know of curves of genus 2, namely as hyperelliptic curves, is in fact the only way that they can occur!