Math 203B: Algebraic Geometry
UCSD, winter 2016, Kiran S. Kedlaya
Modules over schemes

So far, we have attached geometric objects to rings, but not to modules over rings. We remedy that now.

1 Sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. A *(pre)sheaf of modules* on X is a (pre)sheaf of abelian groups $\mathcal{F}$ on X plus a morphism $\mu : \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ of sheaves of abelian groups, such that for each open subset U, the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ provides $\mathcal{F}(U)$ with the structure of a $\mathcal{O}(U)$-module. (For example, μ has to be associative.) A morphism of (pre)sheaves of modules is defined similarly.

For example, if $X = \mathrm{Spec}(R)$ is an affine scheme and M is an R-module, we may define a sheaf $\tilde{M}$ associated to M by analogy with the definition of the structure sheaf: namely, let $\tilde{M}(U)$ be the functions $s : U \to \sqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ with $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ which are locally defined by elements; that is, U is covered by distinguished open subsets $D(f)$ on which we can find elements $m_f \in M_f$ such that for all $\mathfrak{p} \in D(f)$, $s(\mathfrak{p})$ is the image of m_f in $M_{\mathfrak{p}}$. Note that the stalk $\tilde{M}_{\mathfrak{p}}$ equals the localization $M_{\mathfrak{p}}$.

Theorem 1 (Second fundamental theorem of schemes)**.** *For every distinguished open subset $D(f)$ of X, we have $\tilde{M}(D(f)) = M_f$.*

Proof. This again reduces to the case $f = 1$, and then the argument is exactly analogous to the case of the structure sheaf. □

Note that for $M, N \in \mathbf{Mod}_R$, any morphism $\tilde{M} \to \tilde{N}$ of sheaves on $\mathrm{Spec}(R)$ now gives rise to a morphism $M \to N$ of modules by taking global sections, and in fact must be the morphism of sheaves arising from that morphism of modules (since every element of $\tilde{M}$ can locally be written as m/f for some $m \in M, f \in R$, and we know where these have to map). That is, we now have a fully faithful functor from $\mathbf{Mod}_R$ to sheaves of modules on $\mathrm{Spec}(R)$.

2 Quasicoherent sheaves

However, that functor is not essentially surjective; that is, not every sheaf of modules on $\mathrm{Spec}(R)$ arises from an R-module. For example, if R is a discrete valuation ring with fraction field K, then there is a sheaf of modules $\mathcal{F}$ on $\mathrm{Spec}(R) = \{0, \mathfrak{p}\}$ such that

$$\mathcal{F}(\emptyset) = 0, \qquad \mathcal{F}(\{0\}) = K, \qquad \mathcal{F}(\mathrm{Spec}(R)) = 0$$

which does not arise from any R-module.

On the other hand, it is at least clear that every morphism $\tilde{M} \to \mathcal{F}$ is uniquely determined by the map on global sections (again because elements of $\tilde{M}$ are locally m/f). That is, the functor $M \mapsto \tilde{M}$ is left adjoint to the global sections functor on sheaves of modules.

For $(X, \mathcal{O}_X)$ a scheme, we define a *quasicoherent sheaf of modules* on X to be a sheaf of modules $\mathcal{F}$ with the property that for some covering of X by distinguished open affine subspaces $U = \operatorname{Spec}(R)$, each restriction $\mathcal{F}|_U$ is isomorphic to $\tilde{M}$ for some $M \in \mathbf{Mod}_R$; that M has to be $\mathcal{F}(U)$, so this happens if and only if the adjunction map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}$ is an isomorphism. Note that the restriction of a quasicoherent sheaf to an open subspace is again a quasicoherent sheaf.

The obvious difficulty here is that for $X = \operatorname{Spec}(R)$, we may have just defined a larger class of sheaves than the ones coming from modules, since the condition here only applies over some unspecified covering of X.

Theorem 2 (Third fundamental theorem of schemes). *For $X = \operatorname{Spec}(R)$, the functor $M \mapsto \tilde{M}$ is an equivalence of categories between $\mathbf{Mod}_R$ and the category of quasicoherent sheaves on $\operatorname{Spec}(R)$.*

Note that the only missing content in this statement is that if $\mathcal{F}$ is a quasicoherent sheaf, then $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$ is an isomorphism. That is, if we put $M = \mathcal{F}(X)$, then for each $\mathfrak{p} \in \operatorname{Spec}(R)$ the map $M_\mathfrak{p} \to \mathcal{F}_\mathfrak{p}$ should be an isomorphism. (Compare Hartshorne, Lemma II.5.3.)

We first check that the map is injective. By definition, we can cover X with finitely many distinguished opens $U_i = D(f_i) : i = 1, \ldots, n$ such that for $M_i = \mathcal{F}(U_i)$, the map $\tilde{M}_i \to \mathcal{F}|_{U_i}$ is an isomorphism. Suppose $m \in M$ maps to zero in $\mathcal{F}_\mathfrak{p}$; it then maps to zero in $\mathcal{F}(D(f))$ for some $f \in R - \mathfrak{p}$. Then for each i, m maps to zero in $\mathcal{F}(U_i \cap D(f)) = (M_i)_f$, so in M_i it is killed by f^n for some nonnegative integer n. By taking the maximum of n over i, we get a single n such that $f^n m$ maps to zero in M_i; that is, fm is zero as a section of $\mathcal{F}$, and hence as an element of M. Thus m maps to zero in $M_\mathfrak{p}$.

We next check that the map is surjective. It suffices to check that for any $f \in R - \mathfrak{p}$, any section $s \in \mathcal{F}(D(f))$ lifts to M_f. For each i, s lifts to $m_i/f^{n_i} \in (M_i)_f$; we can again take the n_i equal to a single largest value n, but then the differences $m_i - m_j$ need not die in M_{ij}. However, they do get killed by some power of f, so by increasing n suitably we can ensure that the m_i do in fact form an element of

$$\ker\left(\prod_i M_i \to \prod_{i,j} M_{ij}\right) = M.$$

We thus get $m \in M$ such that m/f^n gives rise to s.