So far, we have attached geometric objects to rings, but not to modules over rings. We remedy that now.

1 Sheaves of modules

Let \((X, \mathcal{O}_X)\) be a ringed space. A (pre)sheaf of modules on \(X\) is a (pre)sheaf of abelian groups \(\mathcal{F}\) on \(X\) plus a morphism \(\mu : \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}\) of sheaves of abelian groups, such that for each open subset \(U\), the map \(\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)\) provides \(\mathcal{F}(U)\) with the structure of a \(\mathcal{O}(U)\)-module. (For example, \(\mu\) has to be associative.) A morphism of (pre)sheaves of modules is defined similarly.

For example, if \(X = \text{Spec}(R)\) is an affine scheme and \(M\) is an \(R\)-module, we may define a sheaf \(\tilde{M}\) associated to \(M\) by analogy with the definition of the structure sheaf: namely, let \(\tilde{M}(U)\) be the functions \(s : U \to \bigcup_{p \in U} M_p\) with \(s(p) \in M_p\) which are locally defined by elements; that is, \(U\) is covered by distinguished open subsets \(D(f)\) on which we can find elements \(m_f \in M_f\) such that for all \(p \in D(f)\), \(s(p)\) is the image of \(m_f\) in \(M_p\). Note that the stalk \(\tilde{M}_p\) equals the localization \(M_p\).

**Theorem 1** (Second fundamental theorem of schemes). For every distinguished open subset \(D(f)\) of \(X\), we have \(\tilde{M}(D(f)) = M_f\).

**Proof.** This again reduces to the case \(f = 1\), and then the argument is exactly analogous to the case of the structure sheaf. \(\square\)

Note that for \(M, N \in \text{Mod}_R\), any morphism \(\tilde{M} \to \tilde{N}\) of sheaves on \(\text{Spec}(R)\) now gives rise to a morphism \(M \to N\) of modules by taking global sections, and in fact must be the morphism of sheaves arising from that morphism of modules (since every element of \(\tilde{M}\) can locally be written as \(m/f\) for some \(m \in M, f \in R\), and we know where these have to map). That is, we now have a fully faithful functor from \(\text{Mod}_R\) to sheaves of modules on \(\text{Spec}(R)\).

2 Quasicoherent sheaves

However, that functor is not essentially surjective; that is, not every sheaf of modules on \(\text{Spec}(R)\) arises from an \(R\)-module. For example, if \(R\) is a discrete valuation ring with fraction field \(K\), then there is a sheaf of modules \(\mathcal{F}\) on \(\text{Spec}(R) = \{0, \mathfrak{p}\}\) such that

\[
\mathcal{F}(\emptyset) = 0, \quad \mathcal{F}\{\emptyset\} = K, \quad \mathcal{F}(\text{Spec}(R)) = 0
\]

which does not arise from any \(R\)-module.
On the other hand, it is at least clear that every morphism $\tilde{M} \to F$ is uniquely determined by the map on global sections (again because elements of $\tilde{M}$ are locally $m/f$). That is, the functor $M \mapsto \tilde{M}$ is left adjoint to the global sections functor on sheaves of modules.

For $(X, \mathcal{O}_X)$ a scheme, we define a quasicoherent sheaf of modules on $X$ to be a sheaf of modules $F$ with the property that for some covering of $X$ by distinguished open affine subspaces $U = \text{Spec}(R)$, each restriction $F|_U$ is isomorphic to $\tilde{M}$ for some $M \in \text{Mod}_R$; that $M$ has to be $F(U)$, so this happens if and only if the adjunction map $\tilde{F}(U) \to F$ is an isomorphism. Note that the restriction of a quasicoherent sheaf to an open subspace is again a quasicoherent sheaf.

The obvious difficulty here is that for $X = \text{Spec}(R)$, we may have just defined a larger class of sheaves than the ones coming from modules, since the condition here only applies over some unspecified covering of $X$.

**Theorem 2** (Third fundamental theorem of schemes). For $X = \text{Spec}(R)$, the functor $M \mapsto \tilde{M}$ is an equivalence of categories between $\text{Mod}_R$ and the category of quasicoherent sheaves on $\text{Spec}(R)$.

Note that the only missing content in this statement is that if $F$ is a quasicoherent sheaf, then $\tilde{F}(X) \to F$ is an isomorphism. That is, if we put $M = \tilde{F}(X)$, then for each $p \in \text{Spec}(R)$ the map $M_p \to F_p$ should be an isomorphism. (Compare Hartshorne, Lemma II.5.3.)

We first check that the map is injective. By definition, we can cover $X$ with finitely many distinguished opens $U_i = D(f_i) : i = 1, \ldots, n$ such that for $M_i = \tilde{F}(U_i)$, the map $M_i \to F|_{U_i}$ is an isomorphism. Suppose $m \in M$ maps to zero in $F_p$; it then maps to zero in $F(D(f))$ for some $f \in R - p$. Then for each $i$, $m$ maps to zero in $F(U_i \cap D(f)) = (M_i)_f$, so in $M_i$ it is killed by $f^n$ for some nonnegative integer $n$. By taking the maximum of $n$ over $i$, we get a single $n$ such that $f^nm$ maps to zero in $M_i$; that is, $fm$ is zero as a section of $F$, and hence as an element of $M$. Thus $m$ maps to zero in $M_p$.

We next check that the map is surjective. It suffices to check that for any $f \in R - p$, any section $s \in F(D(f))$ lifts to $M_f$. For each $i$, $s$ lifts to $m_i/f^n \in (M_i)_f$; we can again take the $n_i$ equal to a single largest value $n$, but then the differences $m_i - m_j$ need not die in $M_{ij}$. However, they do get killed by some power of $f$, so by increasing $n$ suitably we can ensure that the $m_i$ do in fact form an element of $M$.

$$\ker \left( \prod_i M_i \to \prod_{i,j} M_{ij} \right) = M.$$ We thus get $m \in M$ such that $m/f^n$ gives rise to $s$.