Math 203B: Algebraic Geometry
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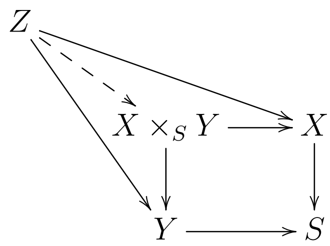
Fiber products

Given two morphisms $X \to S, Y \to S$ in some category, a *fiber product* (or *fibered product*, or *fibre product*, or *fibred product*) of these two morphisms consists of an object $X \times_S Y$ and morphisms $X \times_S Y \to X, X \times_S Y \to Y$ with the following universal property: for any object Z and any morphisms $Z \to X, Z \to Y$ such that the compositions $Z \to X \to S, Z \to Y \to S$ coincide, there is a unique morphism $Z \to X \times_S Y$ such that the compositions $Z \to X \times_S Y \to X, Z \to X \times_S Y \to Y$ coincide with the originally given maps $Z \to X, Z \to Y$. In pictures, any commutative diagram



is completed by some unique choice of the dashed arrow. The goal of this lecture is to prove the following theorem.

Theorem 1. *The category of schemes admits fiber products.*

1 More on fiber products

Before continuing, let us discuss fiber products in some other categories. First, let's look at the category of sets. There, a fiber product of $f : X \to S$, $g : Y \to S$ is given by

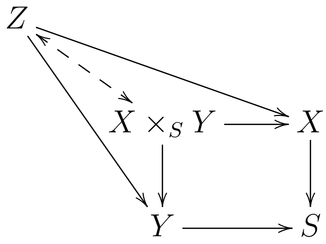
$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the maps $X \times_S Y \to X, X \times_S Y \to Y$ being the compositions of the inclusion $X \times_S Y \to X \times Y$ into the Cartesian product with the standard projections $X \times Y \to X, X \times Y \to Y$.

This construction might look forced, but technically it is not: for instance, I could take

$$X \times_S Y = \{(y, x) \in Y \times X : g(y) = f(x)\}$$

with the maps $X \times_S Y \to X$, $X \times_S Y \to Y$ similarly swapped. This brings up a point that I made in the first lecture: although I will frequently talk about "the" fiber product in a category, any object isomorphic to a fiber product can itself be viewed as a fiber product. What is true is that the fiber product is *unique up to unique isomorphism*: once you fix the object $X \times_S Y$ and the maps to X and Y, if Z is another fiber product equipped with its maps to X and Y, then there is a unique isomorphism providing a choice of the dashed arrow in the diagram



Namely, we get unique maps in each direction using the universal properties, and the compositions in each direction must (by uniqueness of the filling) coincide with the identity map.

Using the fiber product of sets, we can also reformulate the definition of a fiber product in a general category: it should have the property that for any Z,

$$\mathrm{Mor}(Z, X \times_S Y) \to \mathrm{Mor}(Z, X) \times_{\mathrm{Mor}(Z,S)} \mathrm{Mor}(Z, Y)$$

is a bijection. By contrast, an *absolute product* of two objects X, Y is supposed to be an object $X \times Y$ equipped with morphisms $X \times Y \to X, X \times Y \to Y$ such that for any Z,

$$\mathrm{Mor}(Z, X \times Y) \to \mathrm{Mor}(Z, X) \times \mathrm{Mor}(Z, Y)$$

is a bijection. In case the category has a *final object* S (an object to which every object admits a unique morphism), then $X \times Y = X \times_S Y$.

2 Affine schemes

Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} C$ be affine schemes. I claim that $X \times_S Y = \operatorname{Spec}(A \otimes_C B)$ is a fiber product of X and Y over S for the morphisms $X \times_S Y \to X$, $X \times_S Y \to Y$ corresponding to the ring maps $A \to A \otimes_C B, B \to A \otimes_C B$ taking a, b to $a \otimes 1, 1 \otimes b$.

Note that this is clear in the category of affine schemes, because the tensor product is characterized by an analogous universal property with the arrows reversed. In general, the fiber product property is not preserved by enlarging a category, but in this case we are lucky: for Z a general scheme, we have

$$\mathrm{Mor}(Z, X \times_S Y) = \mathrm{Mor}_{\mathbf{Ring}}(A \otimes_C B \to \mathcal{O}_Z(Z))$$

and so forth. Consequently, knowing the fiber product property in the category of affine schemes implies the same in the category of all schemes, and even in the category of locally ringed spaces.

3 Open immersions

We have already noticed that if $(X, \mathcal{O}_X)$ is a scheme and U is an open subset of X, then $(U, \mathcal{O}_X|_U)$ is again a scheme. For the purposes of this discussion, it is convenient to reformulate this construction in an arrow-theoretic fashion: we say that a morphism $f : U \to X$

of locally ringed spaces is an *open immersion* if f induces a homeomorphism of U with an open subset V of X and the induced map $U \to V$ is an isomorphism of locally ringed spaces.

If $X \to S, Y \to S$ are two morphisms of schemes and one of them, say $X \to S$, is an open immersion, then it is easy to construct a fiber product $X \times_S Y$: just take the inverse image of $X \subseteq S$ in Y. Note that the map $X \times_S Y \to Y$ is again an open immersion; this means that the property of being an open immersion is *stable under base change*. We will emphasize this point again a bit later.

In particular, the "intersection" of two open immersions $X \to S, Y \to S$ can be constructed by forming a fiber product $X \times_S Y$.

4 Fiber products in general

Let $\{U_i\}_{i \in I}$ be an open covering of a scheme X. Given morphisms $X \to S, Y \to S$, if we already have constructed fiber products $U_i \times_S Y$, then by above, the maps $(U_i \cap U_j) \times_S Y \to U_i \times_S Y$ are open immersions; so we may glue together the schemes $U_i \times_S Y$ to obtain a fiber product $X \times_S Y$.

Since we already know how to take fiber products when all of X, Y, S are affine, this means we can construct the fiber product $X \times_S Y$ when both S and Y are affine, using an open affine covering of X. We may then turn around and construct it when only S is affine, using an open affine covering of Y.

Now what if S itself is not affine? Choose a covering $\{U_i\}_{i \in I}$ of S by open affines and put $V_i = X \times_S U_i$, $W_i = Y \times_S V_i$ (these are again just the inverse images of U_i in X, Y). We can now construct $V_i \times_{U_i} W_i$ and again glue them together to obtain $X \times_S Y$.

5 Absolute products

By the way, the category of schemes does have a final object, namely $\operatorname{Spec} \mathbb{Z}$. Since

$$\operatorname{Mor}_{\mathbf{Sch}}(X, \operatorname{Spec} \mathbb{Z}) = \operatorname{Mor}_{\mathbf{Ring}}(\mathbb{Z}, \mathcal{O}_X(X)),$$

this comes down to the fact that every ring admits a unique homomorphism from $\mathbb{Z}$, which is obvious.

As a consequence, fiber products over $\operatorname{Spec} \mathbb{Z}$ are in fact absolute products in **Sch**. However, in practice one rarely uses this fact; it is fiber products that are the correct geometric construction. For example, when one thinks of varieties over a field K, they always come with structure morphisms to $\operatorname{Spec} K$, and the analogue of a product of varieties is the fiber product over $\operatorname{Spec} K$.

6 Examples

As noted above, if X, Y are varieties over an algebraically closed field K viewed in the category of schemes, then $X \times_K Y$ is again a variety over K corresponding to the variety-theoretic product. (Note: if A is a ring, I sometimes write $X \times_A Y$ as shorthand for $X \times_{\operatorname{Spec} A} Y$.) So all of your favorite variety-theoretic examples of constructions involving products, like the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$, can be reinterpreted in the language of schemes.

Recall that I defined the projective space $\mathbb{P}^d_R$ over an arbitrary ring, with a structure morphism to $\operatorname{Spec} R$. By glueing, one may build a projective space $\mathbb{P}^d_S$ over any base scheme S. This construction has the following base change property: for any morphism $Y \to X$ of schemes, there is a distinguished isomorphism

$$\mathbb{P}^d_Y \cong \mathbb{P}^d_X \times_X Y.$$

Similarly, if one defines the *affine space* $\mathbb{A}^d_R = \operatorname{Spec} R[x_1, \ldots, x_d]$, then it comes with a map to $\operatorname{Spec} R$, one may define $\mathbb{A}^d_S$ over any base scheme S, and for any morphism $Y \to X$ of schemes, there is a distinguished isomorphism

$$\mathbb{P}^d_Y \cong \mathbb{P}^d_X \times_X Y.$$