Math 203B: Algebraic Geometry
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The Proj construction

As in the case of varieties, we are interested in schemes associated to graded rings, such as projective spaces.

1 Graded rings and homogeneous ideals

A *graded ring* is a ring S equipped with a direct sum splitting of its additive group

$$S = \bigoplus_{n=0}^{\infty} S_n$$

such that $S_m S_n \subseteq S_{m+n}$ for all $m, n \geq 0$. For example, given a ring R, the polynomial ring $S = R[x_0, \dots, x_d]$ may be viewed as a graded ring by taking S_n to be the collection of homogeneous polynomials of total degree n. (But this is not the only way! More on this below.)

For S a graded ring, the subset S_0 is a subring, and the subset

$$S_+ = \bigoplus_{n>0} S_n$$

is an ideal of S. We say a prime ideal $\mathfrak{p}$ of S is *homogeneous* if

$$\mathfrak{p} = \bigoplus_{n=0}^{\infty} (S_n \cap \mathfrak{p}),$$

and *irrelevant* if $S_+ \subseteq \mathfrak{p}$ (otherwise *relevant*). Let $\operatorname{Proj} S \subseteq \operatorname{Spec} S$ be the set of homogeneous relevant prime ideals of S.

We equip $\operatorname{Proj} S$ with the subspace topology from $\operatorname{Spec} S$, i.e., for each ideal I we get a closed subset $V(I)$ consisting of prime ideals containing I. In fact, we only need to use homogeneous ideals to get all the closed subsets: for any ideal I, the ideal generated by the homogeneous components of elements of I defines the same vanishing set in $\operatorname{Proj} S$.

For example, let K be an algebraically closed field and take $S = K[x_0, x_1]$. Which points in $\operatorname{Spec} S$ belong to $\operatorname{Proj} S$?

- The generic point (0) belongs to $\operatorname{Proj} S$.
- By the Nullstellensatz, closed points all have the form $(x - a, y - b)$ for some $a, b \in K$. In order for such a prime ideal to be homogeneous, we must have $a = b = 0$; but (x, y) is irrelevant. So none of these points belong to $\operatorname{Proj} S$.

- All other points have the form (P) where P is an irreducible polynomial. In order for such a prime ideal to be homogeneous, P must be homogeneous; but then it can only be irreducible if it is linear, say $P = ax_0 + bx_1$. We then have points corresponding to the ratios b/a, i.e., the elements of K plus ∞.

As you may have guessed, $\operatorname{Proj} S$ may be canonically identified with $\mathbb{P}^1_K$.

2 Proj as a scheme

To give Proj a scheme structure, we identify *distinguished open subsets* by analogy with the case of affine schemes.

For $f \in S_+$ homogeneous (i.e., $f \in S_d$ for some $d > 0$), define

$$D_+(f) = (\operatorname{Proj} S) \cap D(f) = \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\};$$

this is obviously an open subset of $\operatorname{Proj} S$. Since we only need homogeneous ideals to describe the topology on $\operatorname{Proj} S$, the $D_+(f)$ actually form a basis for the topology.

Recall that we have a distinguished homeomorphism $D(f) \cong \operatorname{Spec} S_f$. In a similar vein, $D_+(f)$ admits a distinguished homeomorphism with $\operatorname{Spec} S_{f,0}$, where

$$S_f = \bigoplus_{n=-\infty}^{\infty} S_{f,n}$$

is the degree decomposition of S_f (with $f \in S_{f,d}$, $f^{-1} \in S_{f,-d}$). Note that f needs to be homogeneous for this definition to make sense.

For $g \in S_+$ also homogeneous of degree d', there is a canonical isomorphism

$$(S_{f,0})_{g^d/f^{d'}} \cong (S_{g,0})_{f^{d'}/g^d},$$

which means that $D_+(fg)$ is a distinguished open subscheme of both $D_+(f)$ and $D_+(g)$. We may thus glue the affine schemes $D_+(f)$ together to provide a scheme structure on $\operatorname{Proj} S$.

As for affine schemes, a *graded homomorphism* of graded rings $S \to S'$ (i.e., one that takes homogeneous elements to homogeneous elements) defines a morphism $\operatorname{Proj} S' \to \operatorname{Proj} S$ of schemes. However, this is nowhere near to defining an equivalence of categories! For example, one can tamper with finitely many of the S_n in positive degrees without changing $\operatorname{Proj} S$; for instance, for any $m > 0$, the ring

$$S_0 \oplus \bigoplus_{n=m}^{\infty} S_n$$

has the same Proj as S itself. More seriously, there is an important geometric invariant derived from the graded ring which is not reflected in the definition of Proj; we will describe it once we have defined quasicoherent sheaves.

The stalk of the structure sheaf of S at a point $\mathfrak{p} \in \operatorname{Proj} S$ can be described as $S_{\mathfrak{p},0}$, where $S_{\mathfrak{p}}$ is again split into its degree decomposition. If you prefer, you may define the structure sheaf directly using maps into the disjoint union of the $S_{\mathfrak{p},0}$; this is done in Hartshorne.

3 Example: weighted projective spaces

For R a ring, viewing $S = R[x_0, \dots, x_d]$ as a graded ring in the usual way, we define the projective space

$$\mathbb{P}^d_R = \operatorname{Proj} S.$$

Note that the ideal S_+ is generated by $x_0, \dots, x_d$, so $\operatorname{Proj} S$ is covered by

$$\begin{aligned} D_+(x_0) &= \operatorname{Spec} R[x_1/x_0, \dots, x_d/x_0], \\ D_+(x_1) &= \operatorname{Spec} R[x_0/x_1, x_2/x_1, \dots, x_d/x_1], \\ &\vdots \\ D_+(x_n) &= \operatorname{Spec} R[x_0/x_d, \dots, x_{d-1}/x_d], \end{aligned}$$

so we have the expected covering by $d+1$ affine subschemes. In particular, the case $d = 1$ agrees with our previous definition.

However, we can also view S as a graded ring in more exotic ways, e.g., by specifying positive integers w_i and declaring that x_i has degree w_i, not 1. This gives rise to spaces called *weighted projective spaces* which are not the same; for example, if K is an algebraically closed field, then $\mathbb{P}^d_K$ is a smooth variety but the weighted projective spaces typically are not (they may have singularities at the points $[1 : 0 : \cdots : 0], \dots, [0 : \cdots : 0 : 1]$).

4 Closed subspaces

Recall that if $R \to R'$ is a surjective morphism of rings, then $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an injective map of topological spaces which is a homeomorphism of $\operatorname{Spec} R'$ with a closed subspace of $\operatorname{Spec} R$ (namely the vanishing set of $\ker(R \to R')$).

Similarly, if $S' \to S$ is a surjective graded morphism of graded rings (i.e., the quotient by a graded ideal), then $\operatorname{Proj} S' \to \operatorname{Proj} S$ is an injective map of topological spaces which is a homeomorphism of $\operatorname{Proj} S'$ with a closed subspace of $\operatorname{Proj} S$ (namely the vanishing set of $\ker(S \to S')$.

This means that all of your favorite projective varieties, e.g., plane curves, arise from the Proj construction. In fact, we would like to say that they are *closed subschemes* of projective spaces... but this is a bit subtle. More on this in an upcoming lecture!