Math 203B (Algebraic Geometry), UCSD, winter 2016
Problem Set 2 (due Wednesday, January 20)

Solve the following problems, and turn in the solutions to *four* of them.

Normally problem sets will be due in class. However, since the next office hours will only occur *after* next Wednesday's lecture, you may turn this problem set in to me or my department mailbox anytime before 5pm on the due date.

1. (a) Give a precise mathematical formulation of the following informal statement: for X a topological space and $\mathcal{B}$ a basis of open subsets of X, the category of sheaves on X is equivalent to the category of "sheaves specified on $\mathcal{B}$".

   (b) Prove the statement you made in (a).

2. Exhibit a morphism in the category of locally ringed spaces from the Riemann sphere, equipped with the sheaf of holomorphic functions, to the scheme $\mathbb{P}^1_{\mathbb{C}}$ which is surjective on closed points. (Optional: show also that this morphism factors through the variety-theoretic $\mathbb{P}^1_{\mathbb{C}}$.)

3. Let X be a scheme and put $A = \mathcal{O}_X(X)$. Let $f_1, \dots, f_n \in A$ be elements which generate the unit ideal. For $i = 1, \dots, n$, let X_i be the open subscheme of X consisting of those points x for which f_i does not belong to the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Suppose that X_i is affine for $i = 1, \dots, n$, and put $A_i = \mathcal{O}_X(X_i)$.

   (a) For $i, j = 1, \dots, n$, prove that the open subscheme $X_i \cap X_j$ of X is isomorphic to $\operatorname{Spec}(A_i[f_j^{-1}])$.

   (b) Prove that the natural map $A_{f_i} \to A_i$ is an isomorphism. Hint: start with an exact sequence

   $$0 \to A \to \bigoplus_{i=1}^{n} A_i \to \bigoplus_{i,j=1}^{n} A_{ij}$$

   for $A_{ij} = \mathcal{O}_X(X_i \cap X_j)$, then invert f_i.

   (c) Prove that X is isomorphic to $\operatorname{Spec}(A)$.

4. Here's a fact we'll use soon in the construction of sheaf cohomology. Let M be a module over a ring R. Let $f_1, \dots, f_n$ be elements of R which generate the unit ideal. We showed in lecture for $M = R$ (but the general case is similar) that there is an exact sequence

   $$0 \to M \to \bigoplus_{i=1}^{n} M_{f_i} \to \bigoplus_{i,j=1}^{n} M_{f_i f_j}.$$

   Show that this extends to an exact sequence

   $$0 \to M \to \bigoplus_{i=1}^{n} M_{f_i} \to \bigoplus_{i,j=1}^{n} M_{f_i f_j} \to \bigoplus_{i,j,k=1}^{n} M_{f_i f_j f_k} \to \cdots,$$

where the definition of the additional terms and maps is left for you to figure out. (It might help to try the case $n = 3$ first, then look for the general pattern.)

5. Let R be a nonzero ring. Prove that the following conditions are equivalent.

   (a) The space $\mathrm{Spec}(R)$ is disconnected: that is, it is the disjoint union of two open-closed proper subsets.

   (b) There exist nonzero elements e_1, e_2 of R with $e_1 + e_2 = 1$ which are *idempotent*, i.e., $e_1^2 = e_1, e_2^2 = e_2$.

   Hint: use the fact that R is isomorphic to the ring of global sections of the structure sheaf.

6. (a) Give an example of a scheme whose closed points are not dense. The Internet can help!

   (b) Show that if $X = \mathrm{Spec}(R)$ where R is a finitely generated algebra over a field, then the closed points of X are dense for the Zariski topology.

7. Let S be the set of morphisms $\mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$. Describe S, and show that in particular $S \neq \mathbb{Z} \cup \{\infty\}$.