Math 203B (Algebraic Geometry), UCSD, winter 2016
Solutions for problem set 3

1. (a) Since f is a unit in R_f, any prime ideal of R_f contracts to a prime ideal of R not containing f. Conversely, if $\mathfrak{p} \in \operatorname{Spec} R$ does not contain f, then $\mathfrak{p}R_f$ is again prime (if $(m_1/f^{n_1})(m_2/f^{n_2}) \in \mathfrak{p}R_f$, then $m_1 m_2 f^? \in \mathfrak{p}$, so $m_1 m_2 \in \mathfrak{p}$) and contracts to $\mathfrak{p}$ (similar argument); hence the map $\operatorname{Spec} R_f \to \operatorname{Spec} R$ has image equal to $D(f)$. The map $\operatorname{Spec} R_f \to D(f)$, which we've just shown is surjective, is also injective: if $\mathfrak{p} \in D(f)$ is the contraction of $\mathfrak{q} \in \operatorname{Spec} R_f$, then $\mathfrak{p}R_f \subseteq \mathfrak{q}$ and $\mathfrak{q} \subseteq \mathfrak{p}R_f$ (both easily). Moreover, the continuous bijection $\operatorname{Spec} R_f \to D(f)$ is continuous: the open subset $D(m/f^?)$ of $\operatorname{Spec} R_f$ coincides with $D(mf) \subseteq D(f)$.

   (b) By definition, $D_+(f)$ is the subset of $\operatorname{Spec} S$ consisting of homogeneous relevant prime ideals not containing f, equipped with the subspace topology. The map $\operatorname{Spec} S_{f,0} \to D_+(f)$ takes $\mathfrak{p} \in \operatorname{Spec} S_{f,0}$ to the ideal $\mathfrak{q} = \bigoplus_{n=0}^\infty \mathfrak{q}_n$ in which $\mathfrak{q}_n$ consists of those $x \in S_n$ such that $x^d f^{-n} \in \mathfrak{p}$. The inverse map takes $\mathfrak{q} = \bigoplus_{n=0}^\infty \mathfrak{q}_n$ to the ideal $\sum_{n=0}^\infty f^{-n}\mathfrak{q}_{dn}$ of $S_{f,0}$. This map is bicontinuous: a basis of the topology on $D_+(f)$ is given by sets of the form $D_+(fg)$ where g is homogeneous of some degree $e > 0$, and these are identified with the sets $D(g^d/f^e) \subseteq \operatorname{Spec} S_{f,0}$ which again form a basis of the topology.

2. The sections of the structure sheaf on $D_+(x_i) \subseteq \mathbb{P}^d_R$ equal the ring $R[x_0/x_i, \dots, x_d/x_i]$, where the factor x_i/x_i is omitted. The global sections are the collections of local sections which agree on overlaps. To compute these, note that the sections on any intersection of the $D_+(x_i)$ inject into the ring $D_+(x_0 \cdots x_d)$, which consists of all formal sums $\sum c_{e_0,\dots,e_d} x_0^{e_0} \cdots x_d^{e_d}$ with $c_{e_0,\dots,e_d} \in R$ and the sum running over all $(d+1)$-tuples $(e_0, \dots, e_d)$ of integers summing to 0. The subring corresponding to $D_+(x_i)$ consists of the sums running over all tuples in which $e_j \geq 0$ for all $j \neq i$. The global sections are given by the intersection of these subring, which consists of sums running over all tuples for which $e_i \geq 0$ for all i. Since $e_0 + \cdots + e_d = 0$, this only happens for the zero tuple, i.e., we just get the ring R. Since $\mathbb{P}^d_R \to \operatorname{Spec} R$ is not an isomorphism (e.g., because it is not injective on underlying sets), $\mathbb{P}^d_R$ cannot be affine.

3. (a) It is obvious that if $\operatorname{Spec} R$ is reduced, then R is reduced. Conversely, if R is reduced, then $R_\mathfrak{p}$ is reduced for each $\mathfrak{p} \in R$: if $(r_1/s_1)^n = 0$ in $R_\mathfrak{p}$, then r_1^n is killed by some $s_2 \in R - \mathfrak{p}$, so $(r_1 s)^n = 0$, so $r_1 s = 0$ because R is reduced, so $r_1/s_1 = 0$ in $R_\mathfrak{p}$. For each open subset U of X, $\mathcal{O}_X(U)$ is a subring of the reduced ring $\prod_{\mathfrak{p} \in R} R_\mathfrak{p}$ and hence is also reduced.

   (b) This is similar to (a). On one hand, if X is reduced, then $\mathcal{O}_{X,x}$ is a direct limit of reduced rings and is hence reduced. Conversely, if $\mathcal{O}_{X,x}$ is reduced for each $x \in X$, then for each open subset U of X, $\mathcal{O}_X(U)$ is a subring of the reduced ring $\prod_{x \in U} \mathcal{O}_{X,x}$ and hence is also reduced.

   (c) For R a ring, the nilpotent elements of R form an ideal (if $x_1^{n_1} = x_2^{n_2} = 0$, then $(x_1 + x_2)^{n_1 + n_2 - 1} = 0$) called the *nilradical* of R; let R_{red} be the quotient of R by the nilradical. Note that $\operatorname{Spec} R_{\text{red}} \to \operatorname{Spec} R$ is a closed immersion which is also surjective on underlying topological spaces (because every prime ideal contains the nilradical), hence a homeomorphism (using the following problem). Also, the functor $R \mapsto R_{\text{red}}$ from rings to reduced rings is left adjoint to the forgetful functor.

For $(X, \mathcal{O}_X)$ a scheme, put $X_{\text{red}} = (X, \mathcal{O}_{X,\text{red}})$, where $\mathcal{O}_{X,\text{red}}(U)$ with the set of functions $s : U \to \sqcup_{x \in X} \mathcal{O}_{X,x,\text{red}}$ with $s(x) \in \mathcal{O}_{X,x,\text{red}}$ for all $x \in U$ which are locally represented by sections of $\mathcal{O}(X)$. It will suffice (both to check that X_{red} is a scheme and to get the adjoint property) to show that if $X = \operatorname{Spec} R$ is affine, then $X_{\text{red}} = \operatorname{Spec} R_{\text{red}}$; this reduces to showing that $\mathcal{O}_{X,\text{red}}(D(f)) = R_{\text{red},f}$. This is true because for $\mathfrak{p} \in \operatorname{Spec} R$, $\mathcal{O}_{X,\mathfrak{p},\text{red}} = (R_{\mathfrak{p}})_{\text{red}} = (R_{\text{red}})_{\mathfrak{p}}$. (By a similar argument, one sees that for every open subset U of X, we have $\mathcal{O}_{X,\text{red}}(U) = \mathcal{O}_X(U)_{\text{red}}$.)

4. The map $\operatorname{Spec} B \to \operatorname{Spec} A$ identifies $\operatorname{Spec} B$ with the set of $\mathfrak{p} \in \operatorname{Spec} A$ containing $I = \ker(A \to B)$. Under this identification, the open subset $D(\overline{f})$ of $\operatorname{Spec} B$ equals $D(f) \cap \operatorname{Spec} B$ for any $f \in A$ lifting $\overline{f} \in B$. Hence $\operatorname{Spec} B$ is homeomorphic to the closed subset $V(I)$ of A. The morphism $f^\sharp : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective on stalks: if $\mathfrak{p} \notin \operatorname{Spec} B$ the target is the zero map, and otherwise it is the map $A_{\mathfrak{p}} \to B_{\mathfrak{p}B}$ which is again surjective.

5. In both cases, it suffices to match up a basis of open subsets. By definition (plus an earlier exercise), $\operatorname{Proj} S$ admits a basis of open subschemes $D_+(f) \cong \operatorname{Spec} S_{f,0}$ for f varying over homogeneous elements of positive degree. The key point is that $D_+(f) = D_+(f^m)$, so we need only consider f of degree divisible by m; for such f, the graded rings S, $S' = S_0 \oplus \bigoplus_{n=m}^\infty S_n$, and $S'' = \bigoplus_{n=0}^\infty S_{mn}$ satisfy $S_{f,0} = S'_{f,0} = S''_{f,0}$ (because they are defined using the same elements).

6. The classical Veronese embedding is defined in terms of homogeneous coordinates by the formula
$$[x : y : z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2].$$
This corresponds to a morphism
$$R[x_0, \dots, x_5] \to R[x, y, z]$$
of graded if we put each of $x_0, \dots, x_5$ in degree 2 rather than degree 1: we may then send $x_0, \dots, x_5$ to $x^2, xy, xz, y^2, yz, z^2$, respectively.