Math 203B (Algebraic Geometry), UCSD, winter 2016
Solutions for problem set 4

1. In one direction, if $\mathcal{F}(\operatorname{Spec} R) = M$ is a finitely generated R-module, then $\mathcal{F}(\operatorname{Spec} R_f) = M_f$ is a finitely generated R-module. In the other direction, suppose that $f_1, \dots, f_n \in R$ generate the unit ideal and $M_{f_1}, \dots, M_{f_n}$ are finitely generated modules over $R_{f_1}, \dots, R_{f_n}$, respectively. We may then choose finite subsets $S_1, \dots, S_n$ of M such that S_i generates M_{f_i} over R_{f_i} (namely, choose a finite set of generators of M_{f_i} and then clear denominators). Put $S = S_1 \cup \cdots \cup S_n$, let F be a free R-module indexed by the elements of S, and let $F \to M$ be the induced morphism. Then for each $\mathfrak{p} \in \operatorname{Spec} R$, there exists $i \in \{1, \dots, n\}$ such that $\mathfrak{p} \in D(f_i)$; then $M_\mathfrak{p} = (M_{f_i})_\mathfrak{p}$ and $F_{f_i} \to M_{f_i}$ is surjective, so $F_\mathfrak{p} \to M_\mathfrak{p}$ is surjective. Since this is true for all $\mathfrak{p}$, it follows that $F \to M$ is surjective, so M is finitely generated. This implies that "$\mathcal{F}(\operatorname{Spec} R)$ is a finitely generated R-module" is a local property, so the affine communication lemma implies that if the property holds for a single covering of X by open affines, then it holds for every open affine subscheme of X.

2. The property of upper semicontinuity may be checked locally on X, so we may assume at once that $X = \operatorname{Spec}(R)$ is affine, so that $\mathcal{F} \cong \tilde{M}$ for $M = \mathcal{F}(X)$. The upper semicontinuity property states that for any $x \in X$, if $\dim_{\kappa(x)} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x = n$, then there exists an open neighborhood U of x in X such that $\dim_{\kappa(y)} \mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y \leq n$ for all $y \in U$. To check this, choose any elements $m_1, \dots, m_n \in \mathcal{F}_x$ which form a basis of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ over $\kappa(x)$. By Nakayama's lemma, $m_1, \dots, m_n$ generate $\mathcal{F}_x$ as a module over $\mathcal{O}_{X,x}$. Now choose some generators $m'_1, \dots, m'_k$ of M as an R-module. In $\mathcal{F}_x$, we can write $m'_i = \sum_j f_{ij} m_j$ for some $f_{ij} \in \mathcal{O}_{X,x}$. Now find an open neighborhood U of x in X such that the m_i, the f_{ij}, and the equality $m'_i = \sum_j f_{ij} m_j$ all lift to U. Then $m_1, \dots, m_n$ generate $\mathcal{F}(U)$, so they also generate $\mathcal{F}_y$ for all $y \in U$. Therefore $\dim_{\kappa(y)} \mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y \leq n$ for all $y \in U$, as desired.

3. (a) An R-module M is *locally free* if there exist $f_1, \dots, f_n \in R$ generating the unit ideal such that M_{f_i} is a free module over R_{f_i}. If M is a locally finite free R-module, then clearly M_f is a locally finite free R_f-module (using the same $f_1, \dots, f_n$). Conversely, suppose that there exist $f_1, \dots, f_n \in R$ generating the unit ideal such that M_{f_i} is a locally finite free module over R_{f_i}. By an earlier exercise, M is then finitely generated, so it suffices to check that it is locally free. By hypothesis, for each i, there exist $g_{i1}, \dots, g_{im} \in R_{f_i}$ generating the unit ideal such that $(M_{f_i})_{g_{ij}}$ is a free module over $(R_{f_i})_{g_{ij}}$. By clearing denominators, we may force $g_{ij} \in R$; we may then identify $(R_{f_i})_{g_{ij}}$ with $R_{f_i g_{ij}}$ and $(M_{f_i})_{g_{ij}}$ with $M_{f_i g_{ij}}$. For each fixed i, the sets $D(f_i g_{ij})$ cover $D(f_i)$; consequently, as both i and j vary, the sets $D(f_i g_{ij})$ cover $\operatorname{Spec} R$. Hence M is locally free.

   (b) The original problem statement was missing some conditions on the vector bundle: it must come with a map $Y \times_X Y \to Y$ corresponding to addition on each $\mathbb{A}^n_{U_i}$, and with a map $\mathbb{A}^1_X \times_X Y \to Y$ corresponding to scalar multiplication on each $\mathbb{A}^n_{U_i}$.

4. It suffices to check that for M a finitely generated module over a reduced ring R such that $\dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p}) = n$ for all $\mathfrak{p} \in \operatorname{Spec} R$, every $\mathfrak{p} \in \operatorname{Spec} R$ admits a distinguished open neighborhood $D(f)$ such that M_f is free of rank n over R_f. Let $m_1, \dots, m_n \in M$ be elements which are linearly independent in $M \otimes_R \kappa(\mathfrak{p})$; they then form a basis. By Nakayama's lemma, they also generate $M_{\mathfrak{p}}$. Now choose some other finite set $s_1, \dots, s_k$ of elements of M which generate M, and choose elements $A_{ij} \in R$ such that $m_j = \sum_i A_{ij} s_i$. The fact that $m_1, \dots, m_n$ are linearly independent in $M_{\mathfrak{p}}$ means that some maximal minor of the matrix A is invertible in $R_{\mathfrak{p}}$; taking f to be this value, we see that for each $\mathfrak{q} \in D(f)$, $m_1, \dots, m_n$ are linearly independent in $M \otimes_R \kappa(\mathfrak{q})$ and hence (by Nakayama again) generate $M_{\mathfrak{q}}$. It follows that the map from the free module R_f^n to M_f defined by $m_1, \dots, m_n$ is surjective. To check that it is injective, choose the coefficients of a relation among $m_1, \dots, m_n$; these project to zero in every prime ideal of R_f, and hence are zero because R is reduced.

5. (a) In one direction, the base change of $\operatorname{Spec} S \to \operatorname{Spec} R$ to $\operatorname{Spec} R_f$ is $\operatorname{Spec} S_f \to \operatorname{Spec} R_f$, where we use the map $R \to S$ to view f as an element of S. In the other direction, suppose that $Y \to \operatorname{Spec} R$ is a morphism and there exist $f_1, \dots, f_n \in R$ such that $Y_i = Y \times_{\operatorname{Spec} R} \operatorname{Spec} R_{f_i}$ is affine. We check that Y is affine by verifying the conditions of HW2 problem 3: the elements $f_1, \dots, f_n \in \mathcal{O}_Y(Y)$ generate the unit ideal (since they do so already in R), and Y_i is the open subscheme of Y consisting of those points y for which $f_i \notin \mathfrak{m}_{Y,y}$ (by the fact that morphisms of schemes induce *local* homomorphisms of local rings).

   (b) Combine (a) with problem 1.

   (c) The fiber of $x \in X$ is equal to the underlying space of the scheme $Y \times_X \operatorname{Spec} \kappa(x)$; so to check that a finite morphism is quasi-finite, we may assume that $X = \operatorname{Spec} K$. But then $Y = \operatorname{Spec} A$ for A a finite K-algebra, and we know that such an algebra has only finitely many connected components (e.g., because each one must have positive dimension).

   For an example of a quasi-finite morphism which is not finite, take the open immersion

$$\operatorname{Spec} K[T, T^{-1}] \to \operatorname{Spec} K[T]$$

   where K is any field. The inverse image of each point is either empty or a single point, but the underlying map of rings is not finite.

6. (a) For Y, X two schemes over some base S, the *graph* of a morphism $f : Y \to X$ of S-schemes is by definition the closed immersion $Y \to Y \times_S X$ corresponding to the pair $(\mathrm{id}_Y : Y \to Y, f : Y \to X)$ (if no base is specified, use the universal base $\operatorname{Spec} \mathbb{Z}$). If L/K is a finite Galois extension of fields with group G, then each $g \in G$ defines a map $L \to L$ of rings over K and hence a morphism $\operatorname{Spec} L \to \operatorname{Spec} L$ of schemes over $\operatorname{Spec} K$, and its graph Γ_g is a closed immersion of $\operatorname{Spec} L$ into $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L$. The claim then is that $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is the disjoint union of open-and-closed subschemes, each of which is the image of one of these

graphs; this amounts to the algebraic statement that $L \otimes_K L$ splits as a direct sum of copies of L. For this, choose a primitive element α for L over K, let $P(T)$ be its minimal polynomial, and write

$$L \otimes_K L \cong L \otimes_K K[T]/(P(T)) \cong L[T]/(P(T)) = \prod L[T]/(T - \alpha_i) \cong \prod L$$

where $P(T) = \prod_i (T - \alpha_i)$.

(b) If L/K is purely inseparable and $L \neq K$, then they are both of some positive characteristic p, and there exists some $x \in L$ which has a p-th root y in L but not in K. Now $y \otimes 1 - 1 \otimes y$ is nonzero in $L \otimes_K L$, but its pth power is $x \otimes 1 - 1 \otimes x = 0$; so $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L = \operatorname{Spec}(L \otimes_K L)$ is not reduced.

7. It suffices to check the claim when Y is affine; in this case, X is itself quasicompact. (Namely, Y is covered by opens whose inverse images are quasicompact, but only finitely many are needed because Y is also quasicompact.) Pick open affine subsets $U_1, \dots, U_n$ which cover X. Because f is quasiseparated, for any i, j, the space $X \times_{X \times_Y X} U_i \times_Y U_j$ is quasicompact, but this space is none other than $U_i \cap U_j$. We can thus choose finitely many open affine subsets V_{ijk} of X that cover $U_i \cap U_j$. Let $\mathcal{F}$ be a quasicoherent sheaf on X; its pushforward is then the sheaf associated to the module which is the kernel of the map

$$\bigoplus_{i=1}^{n} \mathcal{F}(U_i) \to \bigoplus_{i,j=1}^{n} \bigoplus_{k} \mathcal{F}(V_{ijk}).$$