Math 203B: Algebraic Geometry
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Schemes

In the previous lecture, we defined the *structure sheaf* on $\mathrm{Spec}(R)$ for an arbitrary ring R. In this lecture, we describe how to glue this construction together to obtain schemes.

1 Ringed spaces

A *ringed space* is a topological space equipped with a sheaf of rings. This includes the usual suspects: manifolds, C^∞ manifolds, complex varieties, algebraic varieties from last quarter, and affine schemes.

We would like to form a category **RS** of ringed spaces, but for this we need to define the concept of a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces. Clearly this should include a continuous map $f : X \to Y$ of topological spaces, plus some extra relation between $\mathcal{O}_X$ and $\mathcal{O}_Y$. But what should that be?

Our goal is for a ring homomorphism $R \to S$ to give rise to a morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of ringed spaces; this suggests that we should be trying to specify a morphism from $\mathcal{O}_Y$ to $\mathcal{O}_X$. But these are sheaves on different spaces! In order to proceed further, we need to back up and define some constructions that move sheaves from one space to another.

2 Direct and inverse images

Let $f : X \to Y$ be a continuous map of topological spaces. For $\mathcal{F}$ a sheaf (say of sets) on X, define the *direct image* $f_*\mathcal{F}$ to be the presheaf on Y given by the formula

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U));$$

this is easily seen to be a sheaf.

The operation f_* amounts to a *restriction* of $\mathcal{F}$ from the category of sheaves on X to the category of sheaves on Y. Correspondingly, it should come with an associated *promotion* operation in the other direction, which for temporarily opaque reasons we will call f^{-1} rather than f^*. These two operations should form an *adjoint pair*: for $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y, we should have a distinguished isomorphism

$$\mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Mor}_X(\mathcal{G}, f_*\mathcal{F}).$$

This suggests a first guess as to the definition of $f^{-1}\mathcal{G}$: for $U \subseteq X$ an open subset, we would like to take $(f^{-1}\mathcal{G})(U)$ to be $\mathcal{G}(f(U))$. Of course this doesn't make sense because $f(U)$ is not an open set, but we can define it to be the *stalk* of $\mathcal{G}$ at the set $f(U)$, i.e., the direct limit of $\mathcal{G}(V)$ over all open subsets V of Y containing $f(U)$.

So far so good, but this is still only a presheaf, not in general a sheaf. Fortunately, there is a natural way to turn a presheaf into sheaf, called *sheafification*; this is itself the left adjoint of the forgetful functor from sheaves to presheaves. Concretely, given a presheaf $\mathcal{F}$ on X, the sheafification is the sheaf whose values on U is the set of functions $s : U \to \sqcup_{x \in X} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ for each $x \in U$ and s is locally the function associated to a section of $\mathcal{F}$.

3 The category of ringed spaces

We can now define the concept of a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces: it is a continuous map $f : X \to Y$ of locally ringed spaces plus a morphism $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings. Note that by adjointness, we can also interpret $f^\sharp$ as a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Let **RS** be the category of ringed spaces.

For example, let $\varphi : R \to S$ be a morphism of rings, and consider the map $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ of topological spaces. I claim this can be promoted in a natural way to a morphism of locally ringed spaces. To begin with, note that for each $\mathfrak{p} \in \operatorname{Spec}(S)$, φ induces a map $\varphi_\mathfrak{p} : R_{f(\mathfrak{p})} \to S_\mathfrak{p}$. Next, for $U \subset \operatorname{Spec}(R)$ an open set, we must construct a homomorphism $f^\sharp : \mathcal{O}_{\operatorname{Spec}(R)}(U) \to \mathcal{O}_{\operatorname{Spec}(S)}(f^{-1}(U))$ of rings. In fact, the choice is forced: if we view an element of $\mathcal{O}_{\operatorname{Spec}(R)}(U)$ as a certain function $U \to \sqcup_{\mathfrak{p} \in U} R_\mathfrak{p}$, then we may write down a function $f^{-1}(U) \to \sqcup_{\mathfrak{p} \in f^{-1}(U)} S_\mathfrak{p}$ that takes a point $\mathfrak{p} \in f^{-1}(U)$, projects to U, maps to $R_{f(\mathfrak{p})}$, and then applies $\varphi_\mathfrak{p}$. I leave it to you to see that this function indeed belongs to $\mathcal{O}_{\operatorname{Spec}(S)}(f^{-1}(U))$. That gives the map of ringed spaces.

To sum up, there is a natural contravariant functor **Ring** $\to$ **RS** taking R to $\operatorname{Spec}(R)$. This functor is *faithful*: every morphism of rings $R \to S$ is uniquely determined by the corresponding morphism $(\operatorname{Spec}(S), \mathcal{O}_{\operatorname{Spec}(S)}) \to (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$. Namely, for any open subset $U \subseteq \operatorname{Spec}(R)$, $f^\sharp$ induces a ring homomorphism $\mathcal{O}_{\operatorname{Spec}(R)}(U) \to \mathcal{O}_{\operatorname{Spec}(S)}(f^{-1}(U))$; by taking $U = \operatorname{Spec}(R)$ and using the first fundamental theorem of schemes, we get back the map $R \to S$.

However, this functor is not *fully faithful*: not every morphism $(\operatorname{Spec}(S), \mathcal{O}_{\operatorname{Spec}(S)}) \to (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ of ringed spaces is derived from a ring homomorphism $R \to S$ in this fashion. See for instance Hartshorne, Example II.2.3.3.

4 Locally ringed spaces

The problem here is that the definition of a ringed space misses a key feature of all of our standard examples. This is perhaps most obvious in the case of affine schemes: the stalk of the structure sheaf $\mathcal{O}$ at a point $\mathfrak{p} \in \operatorname{Spec}(R)$ is the localization $R_\mathfrak{p}$, which in particular is a *local ring*, a ring with a unique maximal ideal (namely the ideal generated by $\mathfrak{p}$).

Similarly, let $\mathcal{O}$ be the sheaf of continuous real-valued functions on an arbitrary topological space X. For $x \in X$, the stalk $\mathcal{O}_x$ surjects onto $\mathbb{R}$ via evaluation at x. Let I be the kernel of this map. Any element of $\mathcal{O}_x - I$ is represented by a continuous function $f : U \to \mathbb{R}$ for some open neighborhood U of x in X with the property that $f(x) \neq 0$. Since f is continuous, we can find an open neighborhood V of x in U such that $f(y) \neq 0$ for all $y \in V$; consequently, f has a multiplicative inverse in $\mathcal{O}_x$. It follows that $\mathcal{O}_x$ is a local ring with maximal ideal I.

To absorb this observation, we define a *locally ringed space* to be a topological space X equipped with a sheaf of rings $\mathcal{O}$ such that for each $x \in X$, the stalk $\mathcal{O}_x$ is a local ring. A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces will be a special kind of morphism of underlying locally ringed spaces. Given a morphism f of ringed spaces, for $x \in X$, $y \in Y$ with $f(x) = y$, $f^\sharp$ induces a map $\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$ of stalks. Since these are both local rings, it makes sense to insist that this be a *local homomorphism*, i.e., the inverse of the maximal ideal of the target is the maximal ideal of the source (rather than some smaller prime ideal). Let **LRS** denote the resulting category of locally ringed spaces.

In particular, any morphism $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ of ringed spaces coming from a morphism $R \to S$ of rings is also a morphism of locally ringed spaces, because the map $\varphi_{\mathfrak{p}}$ is a local homomorphism. In the other direction, we have the following.

Theorem 1 *The functor* **Ring** $\to$ **LRS** *is fully faithful. That is, for any two rings* R, S*, every morphism* $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ *of locally ringed spaces comes from a homomorphism* $R \to S$ *of rings (which we already know to be unique).*

In fact, an even stronger statement is true.

Theorem 2 *Let* R *be a ring and let* $(X, \mathcal{O}_X)$ *be a locally ringed space. Then the map*

$$\operatorname{Mor}_{\mathbf{LRS}}(X, \operatorname{Spec}(R)) \to \operatorname{Mor}_{\mathbf{Ring}}(R, \mathcal{O}_X(X))$$

obtained by taking global sections is a bijection.

To prove this, we build the functor the other way. We start with the underlying map of sets. For $x \in X$, we are given a map $\varphi : R \to \mathcal{O}_X(X)$, which then maps to the stalk $\mathcal{O}_{X,x}$. The latter is a local ring, so it has a unique maximal ideal $\mathfrak{p}_x$; that ideal contracts to a point $f(x) \in \operatorname{Spec}(R)$. That defines a map $f : X \to \operatorname{Spec}(R)$. (Note that we already used the fact that we are working in **LRS** rather than **RS**.)

We next check that this map is continuous. It suffices to check that every distinguished open subset $D(g) \subseteq \operatorname{Spec}(R)$ has inverse image which is open in X. Suppose $x \in X$ is in this inverse image; this means that in the previous paragraph, $g \notin \varphi^{-1}(\mathfrak{p}_x)$ or equivalently $\varphi(g) \notin \mathfrak{p}_x$. Since $\mathcal{O}_{X,x}$ is a local ring, this means that $\varphi(g)$ is a unit in $\mathcal{O}_{X,x}$, so it lifts to an invertible section of $\mathcal{O}_{X,x}$ on some open neighborhood of x; the latter is then contained in the inverse image of $D(g)$.

We next construct the morphism of ringed spaces; this should be a homomorphism $\mathcal{O}_{\operatorname{Spec}(R)} \to f_*\mathcal{O}_X$. It is enough to specify the effect on sections over a general distinguished open set $D(g)$, i.e., the map $R_g \to \mathcal{O}_X(f^{-1}(D(g)))$. This map is determined by the map $R \to \mathcal{O}_X(X) \to \mathcal{O}_X(f^{-1}(D(g)))$ plus the fact that the image of g in $\mathcal{O}_X(X)$ is invertible in $\mathcal{O}_X(f^{-1}(D(g)))$ (because it is invertible in $\mathcal{O}_{X,x}$ for each $x \in f^{-1}(D(g))$ by the previous paragraph).

We next note that this is also a morphism of locally ringed spaces, because the map $R_{f(x)} \to \mathcal{O}_{X,x}$ is by construction a local homomorphism.

Finally, we need to check that the compositions both ways give the identity. It is easy to see that starting with a morphism in **Ring** and then going to and from **LRS** gives the same initial morphism. If we instead start with a morphism in **LRS** and then cycle back, we may see that we get the same initial morphism by seeing that we end up with the same maps of stalks.

5 Schemes

A *scheme* is a locally ringed space which is covered by open subspaces isomorphic to affine schemes. Note that this implies that every point has a neighborhood basis consisting of affine schemes.

In light of the preceding theorems, there is no need to specially define morphisms of schemes: we simply take them to be morphisms of the underlying locally ringed spaces! This simplifies the theory considerably.

As an example, we define the projective line over an arbitrary ring R. Consider the affine schemes

$$U_1 = \operatorname{Spec} R[t], \qquad U_2 = \operatorname{Spec} R[t^{-1}], \qquad U_3 = \operatorname{Spec} R[t, t^{-1}];$$

then U_3 is isomorphic to the distinguished open subset $D(t)$ of U_1 and with the distinguished open subset $D(t^{-1})$ of U_2. Let $\mathbb{P}^1_R$ be the union of U_1 and U_2 along U_3; then $\mathbb{P}^1_R$ inherits the structure of a scheme. Note that the difference $\mathbb{P}^1_R - U_1$ consists of the subspace of U_2 corresponding to $V(t^{-1})$; this is a whole copy of $\operatorname{Spec} R$, not just a single point. This corresponds to the geometric intuition that $\mathbb{P}^1_R$ is actually a *family* of projective spaces indexed by the points of $\operatorname{Spec}(R)$; we'll add more content to that observation later. One point to observe now is that by the above theorem, the maps $R \to R[t], R \to R[t^{-1}], R \to R[t, t^{-1}]$ give rise to a morphism of schemes $\mathbb{P}^1_R \to \operatorname{Spec}(R)$, called the *structure morphism.*

6 Varieties and schemes

Let K be an algebraically closed field. Let R be a reduced finitely generated K-algebra. Let $\operatorname{Maxspec}(R)$ denote the set of maximal ideals of R; by the Nullstellensatz, these all have residue field K. There is an obvious inclusion $j : \operatorname{Maxspec}(R) \subseteq \operatorname{Spec}(R)$ with dense image; the *Zariski topology* on $\operatorname{Maxspec}(R)$ is just the subspace topology for this inclusion.

Last quarter, we constructed a sheaf of *regular functions* on $\operatorname{Maxspec}(R)$, giving the latter the structure of an *affine variety.* This is just $j^{-1}\mathcal{O}_{\operatorname{Spec}(R)}$, so j gives rise to a morphism of locally ringed spaces.

A *variety* over K is a locally ringed space covered by open subspaces which are affine varieties over K. Well, almost: every affine variety as defined above comes with a map $\operatorname{Maxspec}(R) \to \operatorname{Spec}(K)$, and we insist that a variety also come with such a morphism.

Similarly, a *scheme over* K is a scheme plus a morphism to $\operatorname{Spec}(K)$. These form a category in which morphisms must commute with the maps to $\operatorname{Spec}(K)$. It is easily checked (see Hartshorne, Proposition II.2.6) that there is a fully faithful functor from varieties over

K to schemes over K. Just as above, if W is a variety and X is the corresponding scheme, then there is a distinguished morphism $W \to X$ of locally ringed spaces which restricts to a bijection of W to the set of closed points of X.