Math 203B: Algebraic Geometry
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Cohomology of quasicoherent sheaves on projective spaces

We now focus on quasicoherent sheaves on projective spaces.

1 A note on closed subschemes

Suppose for a moment that $j : X \to W$ is a closed immersion. We have already discussed the fact that if $\mathcal{F}$ is a quasicoherent sheaf on X, then $j_* \mathcal{F}$ is a quasicoherent sheaf on W. In fact, we can say a bit more.

Theorem 1. *(a) The functor j_* defines an equivalence of categories between quasicoherent sheaves on X and quasicoherent sheaves on W annihilated by $\mathcal{I} = \ker(\mathcal{O}_W \to f_* \mathcal{O}_X)$.*

(b) For $\mathcal{F}$ a quasicoherent sheaf on W, the groups $H^i(X, \mathcal{F})$ and $H^i(W, j_* \mathcal{F})$ are canonically isomorphic for all i.

Proof. Based on previous statements, both results are easily seen to be true if W is affine. From this, one may deduce a comparison for Čech cohomology groups, and then for sheaf cohomology using spectral sequences. □

For this reason, statements I make about $\mathbb{P}^d_R$ will typically have immediate consequences for closed subschemes of $\mathbb{P}^d_R$.

2 A useful exact sequence

When looking at cohomology on $\mathbb{P}^d_R$, we will frequently use the following observation related to the previous one. Let H be a hyperplane in $X = \mathbb{P}^d_R$, e.g., the zero locus of x_i for some $i \in \{0, \dots, d\}$. Then for $j : H \to \mathbb{P}^d_R$ the corresponding closed immersion, we have an exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to j_* \mathcal{O}_H \to 0.$$

Twisting, we get exact sequences

$$0 \to \mathcal{O}_X(n-1) \to \mathcal{O}_X(n) \to j_* \mathcal{O}_H(n) \to 0.$$

(In fact, this sequence is easier to see when $n \gg 0$.)

3 Cohomology of twisting sheaves: the case $d = 1$

For $\mathcal{F}$ a quasicoherent sheaf on $\mathbb{P}^1_R$, we can compute $H^0(\mathbb{P}^1_R, \mathcal{F})$ and $H^1(\mathbb{P}^1_R, \mathcal{F})$ as the kernel and cokernel of the map

$$\mathcal{F}(D_+(x_0)) \times \mathcal{F}(D_+(x_1)) \to \mathcal{F}(D_+(x_0 x_1)), \qquad (s_0, s_1) \mapsto s_0 - s_1,$$

and $H^i(\mathbb{P}^1_R, \mathcal{F}) = 0$ for all $i > 0$. Let's do this for $\mathcal{F} = \mathcal{O}(n)$, in which case the map is

$$x_0^n R\left[\frac{x_1}{x_0}\right] \oplus x_1^n R\left[\frac{x_0}{x_1}\right] \to x_0^n R\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}\right].$$

We already have observed that the kernel is the set of homogeneous polynomials of degreee n in x_0, x_1, which is nonzero if and only if $n \geq 0$. As for the cokernel, we can write it as $x_0^{-1}x_1^{-1}$ times the set of homogeneous polynomials of degree $n-2$ in x_0^{-1}, x_1^{-1}; in particular, it is nonzero if and only if $n \leq -2$. To summarize, $H^0(X, \mathcal{O}(n))$ and $H^1(X, \mathcal{O}(n))$ are free modules of respective ranks $\max\{n+1, 0\}$ and $\max\{1-n, 0\}$.

4 Cohomology of twisting sheaves: the general case

Let's again take $\mathcal{F} = \mathcal{O}(n)$ but now $X = \mathbb{P}^d_R$ with $d \geq 1$ arbitrary. We again know that $H^0(X, \mathcal{O}(n))$ is the set of homogeneous polynomials of degree n, which is zero if $n < 0$ and otherwise is free of rank $\binom{n+d}{d}$. (Fast way to remember this binomial coefficient: write each monomial as $x_0 \cdots x_0 \times x_1 \cdots \times x_1 \cdots$, then note that the monomials correspond to the choice of the positions of d multiplication signs in a string of length $n+d$.)

Meanwhile, $H^d(\mathbb{P}^d_R, \mathcal{O}(n))$ is the cokernel of the map

$$\prod_{i=0}^{d} \mathcal{O}(n)(D_+(x_0 \cdots \widehat{x_i} \cdots x_d)) \to \mathcal{O}(n)(D_+(x_0 \cdots x_d)),$$

and can be written as $x_0^{-1} \dots x_d^{-1}$ times the set of homogeneous polynomials of degree $n-d-1$ in $x_0^{-1}, \dots, x_d^{-1}$. In particular, it is zero if $n \geq -d$ and otherwise is free of rank $\binom{-n-1}{d}$.

What about the terms in the middle? We claim that $H^i(\mathbb{P}^d_R, \mathcal{O}(n)) = 0$ for all $0 < i < d$ and all n. We'll complete the proof of this later; for the moment, let me illustrate the case $d = 2$ using the exact sequence

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^2_R, \mathcal{O}(n-1)) \to H^0(\mathbb{P}^2_R, \mathcal{O}(n)) \to H^0(\mathbb{P}^1_R, \mathcal{O}(n)) \\ &\to H^1(\mathbb{P}^2_R, \mathcal{O}(n-1)) \to H^1(\mathbb{P}^2_R, \mathcal{O}(n)) \to H^1(\mathbb{P}^1_R, \mathcal{O}(n)) \\ &\to H^2(\mathbb{P}^2_R, \mathcal{O}(n-1)) \to H^2(\mathbb{P}^2_R, \mathcal{O}(n)) \to 0. \end{aligned}$$

For $n \geq 0$, this sequence truncates to

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^2_R, \mathcal{O}(n-1)) \to H^0(\mathbb{P}^2_R, \mathcal{O}(n)) \to H^0(\mathbb{P}^1_R, \mathcal{O}(n)) \\ &\to H^1(\mathbb{P}^2_R, \mathcal{O}(n-1)) \to H^1(\mathbb{P}^2_R, \mathcal{O}(n)) \to 0, \end{aligned}$$

and by inspection we see that $H^0(\mathbb{P}^2_R, \mathcal{O}(n)) \to H^0(\mathbb{P}^1_R, \mathcal{O}(n))$ is surjective, so we deduce that $H^1(\mathbb{P}^2_R, \mathcal{O}(n-1)) \cong H^1(\mathbb{P}^2_R, \mathcal{O}(n))$. For $n \leq 0$, a similar truncation on the other end yields $H^1(\mathbb{P}^2_R, \mathcal{O}(n-1)) \cong H^1(\mathbb{P}^2_R, \mathcal{O}(n))$.

In other words, if we take the sheaf $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(n)$, then multiplication by x_2 defines a bijection on $H^1(\mathbb{P}^2_R, \mathcal{F})$. Now view $S = \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^2_R, \mathcal{F})$ as a module over $R[x_0, x_1, x_2]$.

On one hand, multiplication by x_2 on S is bijective. On the other hand, if we localize at x_2, we get (a bunch of copies of) the cohomology of $\mathcal{F}$ on $D_+(x_i)$ computed using the covering $D_+(x_0x_2), D_+(x_1x_2), D_+(x_2)$; and we know that's zero because $D_+(x_i)$ is an affine scheme. These two facts combine to imply that $S = 0$, whence all of the $H^1(\mathbb{P}^2_R, \mathcal{O}(n))$ vanish.

5 Applications to coherent sheaves

All of these are due to Serre.

Theorem 2. *Let $j : X \to \mathbb{P}^d_R$ be a closed immersion. Let $\mathcal{F}$ be a coherent (quasicoherent locally finitely generated) sheaf on X. Then there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$ and all $i > 0$, $H^i(X, \mathcal{F}(n)) = 0$.*

Proof. By replacing $\mathcal{F}$ with $j_*\mathcal{F}$, we reduce immediately to the case $X = \mathbb{P}^d_R$. We proceed by descending induction on i, noting that the claim is automatic if $i > d$ since we can compute sheaf cohomology using the Čech complex corresponding to the cover by $D_+(x_0), \dots, D_+(x_d)$.

By the previous theorem of Serre, we can find an index $n_1 \in \mathbb{Z}$ such that $\mathcal{F}(n_1)$ is generated by finitely many global sections; that is, we can write down an exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}^{\oplus k} \to \mathcal{F}(n_1) \to 0$$

for some $k \geq 0$. Twisting, we obtain

$$0 \to \mathcal{G}(n - n_1) \to \mathcal{O}(n - n_1)^{\oplus k} \to \mathcal{F}(n) \to 0$$

In the resulting exact sequence

$$H^i(\mathbb{P}^d_R, \mathcal{O}(n - n_1)^{\oplus k}) \to H^i(\mathbb{P}^d_R, \mathcal{F}(n)) \to H^{i+1}(\mathbb{P}^d_R, \mathcal{G}(n - n_1)) \to H^{i+1}(\mathbb{P}^d_R, \mathcal{O}(n - n_1)^{\oplus k})$$

the outside terms vanish as soon as $n - n_1 \geq -d$ (this is only really at issue when $i + 1 = d$). Meanwhile, by the induction hypothesis, $H^{i+1}(\mathbb{P}^d_R, \mathcal{G}(n-n_1))$ vanishes for n sufficiently large; it follows that $H^i(\mathbb{P}^d_R, \mathcal{F}(n))$ also vanishes for i sufficiently large. □

Corollary 3. *Let $j : X \to \mathbb{P}^d_R$ be a closed immersion. Let $\mathcal{F}$ be a coherent (quasicoherent locally finitely generated) sheaf on X. Then for any exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves of modules, there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$ and all $i > 0$,

$$0 \to H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n)) \to H^0(X, \mathcal{H}(n)) \to 0$$

is exact.

Theorem 4. *Suppose that the ring R is noetherian. Let $j : X \to \mathbb{P}^d_R$ be a closed immersion. Let $\mathcal{F}$ be a coherent (quasicoherent locally finitely generated) sheaf on X. Then the groups $H^i(X, \mathcal{F})$ are finitely generated R-modules for all $i \geq 0$.*

Proof. Yet again, we reduce to the case $X = \mathbb{P}^d_R$. Again, we operate by descending induction on i, the case $i > d$ being clear. For some n, there exists an exact sequence

$$0 \to \mathcal{G}(-n) \to \mathcal{O}(-n)^{\oplus k} \to \mathcal{F} \to 0.$$

In the resulting exact sequence

$$H^i(\mathbb{P}^d_R, \mathcal{O}(-n)^{\oplus k}) \to H^i(\mathbb{P}^d_R, \mathcal{F}) \to H^{i+1}(\mathbb{P}^d_R, \mathcal{G}(-n)) \to H^{i+1}(\mathbb{P}^d_R, \mathcal{O}^{\oplus k}),$$

the outside terms are finitely generated R-modules by calculation, while $H^{i+1}(\mathbb{P}^d_R, \mathcal{G}(-n))$ is a finitely generated R-module by the induction hypothesis. It follows that $H^i(\mathbb{P}^d_R, \mathcal{F})$ is a finitely generated R-module, completing the induction. $\square$