Math 204B (Algebraic Number theory), UCSD, winter 2015
Problem Set 2 (due Wednesday, January 21)

Solve the following problems, and turn in the solutions to *four* of them. You may use SageMathCloud to submit the entire assignment if you choose. (From now on, I will omit the disclaimer about collaborations, but it will apply for the rest of the term.)

Clarification: you are permitted to cite the result of any given problem in your solution to any subsequent problem, on the same problem set or a later one, regardless of whether or not you submitted a solution to the original problem.

Throughout this problem set, let K be a field complete with respect to a nonarchimedean absolute value, and let $\mathfrak{o}_K$ be the associated valuation ring.

1. Neukirch, exercises II.2.8 and II.2.9 (these together count as one problem).

2. Neukirch, exercises II.4.2 and II.4.3 (these together count as one problem).

3. Neukirch, exercise II.4.1: prove that an infinite algebraic extension of K, when equipped with the unique extension of the absolute value on K, is *never* complete. (Hint: any finite subextension is complete and hence closed.)

4. (a) Prove that $\mathfrak{o}_K$ is integrally closed, that is, any element of K which is a root of a monic polynomial over $\mathfrak{o}_K$ must itself belong to $\mathfrak{o}_K$.

   (b) Suppose that K is a finite extension of $\mathbb{Q}_p$. Prove that $\mathfrak{o}_K$ equals the integral closure of $\mathbb{Z}_p$ in K.

5. Let p be a prime.

   (a) Prove that if $K = \mathbb{Q}_p(\zeta_n)$ for some positive integer n, then $\mathfrak{o}_K = \mathbb{Z}_p[\zeta_n]$.

   (b) Prove that the maximal ideal of $\mathbb{Z}_p[\zeta_p]$ is generated by $\zeta_p - 1$.

   (c) Prove that $\mathbb{Q}_p(\zeta_p)$ contains an element π satisfying $\pi^{p-1} = -p$.

6. For $f = \sum_{n=0}^{\infty} f_n T^n$ a power series in T with coefficients in K, we define the *radius of convergence* of f as
$$\rho(f) = \liminf_{n\to\infty} |f_n|^{-1/n}.$$
   Prove that this has the expected property: for $x \in K$, $\sum_{n=0}^{\infty} f_n x^n$ converges if $|x| < \rho(f)$ and diverges if $|x| > \rho(f)$. Also give examples to show that the series can either converge or diverge in case $|x| = \rho(f)$.

7. See SageMathCloud for this problem.