Math 204B: Number Theory
UCSD, winter 2017
Problem Set 1 (due Wednesday, January 18)

1. Explain, then prove, the following statement: "The completion of $\mathbb{Q}(i)$ with respect to the $(2+i)$-adic absolute value is isomorphic to $\mathbb{Q}_5$."

2. Let p be a prime number.

   (a) Determine which elements of $\mathbb{Q}_p$ are p-th powers. Beware that the case $p = 2$ is a bit different.

   (b) For n a positive integer congruent to 1 mod p, determine which elements of $\mathbb{Q}_p$ are n-th powers.

   (c) Using (a) and (b), prove that the field $\mathbb{Q}_p$ has no nontrivial automorphisms. The point is to show that every automorphism is continuous with respect to the p-adic absolute value. (Hint: first show that every automorphism also acts on $\mathbb{Z}_p$.)

3. Prove that for p and q two distinct primes, the fields $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic. Hint: one approach is to use the previous exercise.

4. Prove that for $p > 2$, for every $x \in p\mathbb{Z}_p$ the series $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges in $\mathbb{Q}_p$. Then explain what happens for $p = 2$.