NOTES ON ISOCRYSALS

KIRAN S. KEDLAYA

Abstract. For varieties over a perfect field of characteristic $p$, étale cohomology with $\mathbb{Q}_\ell$-coefficients is a Weil cohomology theory only when $\ell \neq p$; the corresponding role for $\ell = p$ is played by Berthelot’s rigid cohomology. In that theory, the coefficient objects analogous to lisse $\ell$-adic sheaves are the overconvergent $F$-isocrystals. This expository article is a brief user’s guide for these objects, including some features shared with $\ell$-adic cohomology (purity, weights) and some features exclusive to the $p$-adic case (Newton polygons, convergence and overconvergence). The relationship between the two cases, via the theory of companions, will be treated in a sequel paper.

1. Introduction

Let $k$ be a perfect field of characteristic $p > 0$. For each prime $\ell \neq p$, étale cohomology with $\mathbb{Q}_\ell$-coefficients constitutes a Weil cohomology theory for varieties over $k$, in which the coefficient objects of locally constant rank are the smooth (lisse) $\mathbb{Q}_\ell$-local systems; when $k$ is finite, one also considers lisse Weil $\mathbb{Q}_\ell$-sheaves. This article is a brief user’s guide for the $p$-adic analogues of these constructions; we focus on basic intuition and statements of theorems, omitting essentially all proofs (except for a couple of undocumented variants of existing proofs, which we record in an appendix).

To obtain a Weil cohomology with $p$-adic coefficients, Berthelot defined the theory of rigid cohomology. One tricky aspect of rigid cohomology is that it includes not one, but two analogues of the category of smooth $\ell$-adic sheaves: the category of convergent $F$-isocrystals and the subcategory of overconvergent $F$-isocrystals. The former category can be interpreted in terms of crystalline cohomology (see Theorem 2.2), but the latter can only be described using analytic geometry. (We will implicitly use rigid analytic geometry, but any of the other flavors of analytic geometry over nonarchimedean fields can be equally well used instead.)

The distinction between convergent and overconvergent $F$-isocrystals carries important functional load: overconvergent $F$-isocrystals seem to be the objects which are “classically motivic” whereas convergent $F$-isocrystals can arise from geometric constructions exclusive to characteristic $p$. For example, the “crystalline companion” (in the sense of [20, Conjecture 1.2.10]) to a compatible system of lisse Weil $\mathbb{Q}_\ell$-sheaves is an overconvergent $F$-isocrystal, but in the category of convergent $F$-isocrystals such an object often acquires a nontrivial slope filtration; a typical example is provided by the universal family of elliptic curves (Example 4.5).
When transporting arguments from $\ell$-adic to $p$-adic cohomology, one can often assign the role of $\mathbb{Q}_\ell$-local systems appropriately to either convergent or overconvergent $F$-isocrystals. In a few cases, one runs into difficulties because neither category seems to provide the needed features; on the other hand, in some cases the rich interplay between the constructions makes it possible to transport statements back to the $\ell$-adic side which do not seem to have any direct proof there.

One can continue the story by describing links between $\ell$-adic and $p$-adic coefficients via the theory of companions as alluded to above. However, this would require setting aside the premise of a purely expository paper, as some new results would be required. We have thus chosen to defer this discussion to a sequel paper [58].

**Notation 1.1.** Throughout this paper, let $k$ denote a perfect field of characteristic $p > 0$ (as above), and let $X$ denote a smooth variety over $k$. By convention, we require varieties to be reduced separated schemes of finite type over $k$, but they need not be irreducible. Let $K$ denote the fraction field of the ring of $p$-typical Witt vectors $W(k)$.

### 2. The Basic Constructions

We begin by illustrating the construction of convergent and overconvergent $F$-isocrystals on smooth varieties, following Berthelot’s original approach to rigid cohomology in which the constructions are fairly explicit but not overtly functorial. A more functorial approach, using suitably constructed sites, is described in [62], to which we defer for justification of all unproved claims (and for treatment of nonsmooth varieties).

We will use without comment the fact that coherent sheaves on affinoid spaces correspond to finitely generated modules over the ring of global sections (i.e., Kiehl’s theorem in rigid analytic geometry).

**Definition 2.1.** For $X$ affine, we construct the category $\mathbf{F-Isoc}(X)$ of convergent $F$-isocrystals on $X$ as follows. Using a lifting construction of Elkik [25] (or its generalization by Arabia [6]), we can find a smooth affine formal scheme $P$ over $W(k)$ with special fiber $X$ and a lift $\sigma : P \rightarrow P$ of the absolute Frobenius on $X$. Let $P_K$ denote the Raynaud generic fiber of $P$ as a rigid analytic space over $K$. Then an object of $\mathbf{F-Isoc}(X)$ is a vector bundle $\mathcal{E}$ on $P_K$ equipped with an integrable connection (i.e., an $\mathcal{O}_X$-coherent $\mathcal{D}$-module) and an isomorphism $\sigma^*\mathcal{E} \cong \mathcal{E}$ of $\mathcal{D}$-modules (which we view as a semilinear action of $\sigma$ on $\mathcal{E}$); a morphism in $\mathbf{F-Isoc}(X)$ is a $\sigma$-equivariant morphism of $\mathcal{D}$-modules.

One checks as in [62] (by comparing to a more functorial definition) that the functor $\mathbf{F-Isoc}$ is a stack for the Zariski and étale topologies on $X$. This leads to a definition of $\mathbf{F-Isoc}(X)$ for arbitrary $X$. When $X = \text{Spec } R$ is affine, we will occasionally write $\mathbf{F-Isoc}(\text{Spec } R)$ instead of $\mathbf{F-Isoc}($Spec $R)$.

**Theorem 2.2** (Ogus). The category $\mathbf{F-Isoc}(X)$ is canonically equivalent to the isogeny category associated to the category of crystals of finite $\mathcal{O}_X,$crys-modules equipped with $F$-actions (i.e., isomorphisms with their $F$-pullbacks).

**Proof.** The functor from crystals to $\mathbf{F-Isoc}(X)$ is exhibited in [67] and shown therein to be fully faithful. For essential surjectivity, see [8, Théorème 2.4.2]. □

**Definition 2.3.** For $X \rightarrow Y$ an open immersion of $k$-varieties with $X$ and $Y$ affine (but $Y$ not necessarily smooth), we construct the category $\mathbf{F-Isoc}(X,Y)$ of isocrystals on $X$...
overconvergent within $Y$ as follows. Again using the results of Elkik or Arabia, we can find an affine formal scheme $P$ over $W(k)$ with special fiber $Y$ which is smooth in a neighborhood of $X$ and a lift $\sigma : Q \to Q$ of absolute Frobenius, for $Q$ the open subscheme of $P$ supported on $Y$, which extends to a neighborhood of $Q_{K}$ in $P_{K}$ for the Berkovich topology (i.e., in more classical terminology, a strict neighborhood of $Q_{K}$ in $P_{K}$). Then an object of $\mathbf{F-Isoc}(X, Y)$ is a vector bundle $E$ on some strict neighborhood equipped with an integrable connection and an isomorphism $\sigma^{*}E \cong E$ of $\mathcal{D}$-modules; a morphism in $\mathbf{F-Isoc}(X, Y)$ is a $\sigma$-equivariant morphism of $\mathcal{D}$-modules defined on some strict neighborhood of $Q_{K}$, with two morphisms considered equal if they agree on some (hence any) strict neighborhood on which they are both defined. In particular, restriction of a bundle from one strict neighborhood to another is an isomorphism in $\mathbf{F-Isoc}(X, Y)$.

One again checks as in [62] that the functor $\mathbf{F-Isoc}$ is a stack for the Zariski and étale topologies on $Y$. This leads to a definition of $\mathbf{F-Isoc}(X, Y)$ for an arbitrary open immersion $X \to Y$.

**Remark 2.4.** Given a commutative diagram

$$
\begin{array}{ccc}
X' & \to & Y' \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}
$$

in which $X' \to Y'$ is again an open immersion of $k$-varieties with $X'$ smooth, one obtains a pullback functor $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(X', Y')$. If $X' = X$, then this pullback functor is obviously faithful; we will see later that it is also full (Theorem 5.5).

**Lemma 2.5** (Berthelot). Let $f : Y' \to Y$ be a proper morphism such that $f^{-1}(X) \to X$ is an isomorphism. Then the pullback functor $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(X, Y')$ is an equivalence of categories.

**Proof.** The original but unpublished reference is [8, Théorème 2.3.5]. An alternate reference is [62, Théorème 7.1.8]. \qed

**Definition 2.6.** We define the category $\mathbf{F-Isoc}^{\dagger}(X)$ of overconvergent $F$-isocrystals on $X$ to be $\mathbf{F-Isoc}(X, Y)$ for some (hence any, by Lemma 2.5) open immersion $X \to Y$ with $Y$ a proper $k$-variety. In particular, if $X$ itself is proper, then $\mathbf{F-Isoc}^{\dagger}(X) = \mathbf{F-Isoc}(X)$; in general, $\mathbf{F-Isoc}^{\dagger}(X)$ is a stack for the Zariski and étale topologies on $X$.

**Remark 2.7.** Retain notation as in Remark 2.4. If $X' \to X$ is finite étale of constant degree $d > 0$ and $Y' \to Y$ is finite flat, one also obtains a pushforward functor $\mathbf{F-Isoc}(X', Y') \to \mathbf{F-Isoc}(X, Y)$ such that $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(X', Y') \to \mathbf{F-Isoc}(X, Y)$ is the functor $E \mapsto E^{\boxtimes d}$. In particular, if $X' \to X$ is finite étale, we obtain a pushforward functor $\mathbf{F-Isoc}^{\dagger}(X') \to \mathbf{F-Isoc}^{\dagger}(X)$.

**Remark 2.8.** The pushforward functoriality of $\mathbf{F-Isoc}$ is often used in conjunction with the following observation (a higher-dimensional analogue of Belyi’s theorem in positive characteristic): any projective variety over $k$ of pure dimension $n$ admits a finite morphism to $\mathbb{P}_{k}^{n}$ which is étale over $\mathbb{A}_{k}^{n}$ [41]. Moreover, any given zero-dimensional subscheme of the smooth locus may be forced into the inverse image of $\mathbb{A}_{k}^{n}$; in particular, the smooth locus is covered by open subsets which are finite étale over $\mathbb{A}_{k}^{n}$ (via various maps).
Remark 2.9. Let $\phi : K \to K$ be the Witt vector Frobenius. In case $X = \text{Spec} \ k$, the categories $\mathbf{F-Isoc}(X)$ and $\mathbf{F-Isoc}^\dagger(X)$ coincide, and may be described concretely as the category of finite-dimensional $K$-vector spaces equipped with isomorphisms with their $\phi$-pullbacks.

In general, choose any closed point $x \in X$ with residue field $\ell$ and put $L = \text{Frac} W(\ell)$. Then the pullback functors $\mathbf{F-Isoc}(X) \to \mathbf{F-Isoc}(x), \mathbf{F-Isoc}^\dagger(X) \to \mathbf{F-Isoc}^\dagger(x)$ define fiber functors in $L$-vector spaces; however, these are not neutral fiber functors unless $\ell = \mathbb{F}_p$. For more on the Tannakian aspects of the categories $\mathbf{F-Isoc}(X)$ and $\mathbf{F-Isoc}^\dagger(X)$, see [14].

Much of the basic analysis of convergent and overconvergent $F$-isocrystals involves “local models” of the global statements under consideration. We describe the basic setup using notation as in [17].

Remark 2.10. Put $\Omega = W(k)[t]$. Let $\Gamma$ be the $p$-adic completion of $W(k)((t))$. Let $\Gamma_c$ be the subring of $\Gamma$ consisting of Laurent series convergent in some region of the form $* \leq |t| < 1$. Each of these rings carries a Frobenius lift $\sigma$ with $\sigma(t) = t^p$ and a derivation $\frac{d}{dt}$.

Define the categories

$$
\mathbf{F-Isoc}(k[t]), \mathbf{F-Isoc}(k((t))), \mathbf{F-Isoc}^\dagger(k((t)))
$$

to consist of finite projective modules over the respective rings $\Omega[p^{-1}], \Gamma[p^{-1}], \Gamma_c[p^{-1}]$ equipped with compatible actions of $\sigma$ and $\frac{d}{dt}$. Here compatibility means that the commutation relation between $\sigma$ and $\frac{d}{dt}$ on the modules is the same as on the base ring:

$$
\frac{d}{dt} \circ \sigma = pt^{p-1}\sigma \circ \frac{d}{dt}.
$$

For some purposes, it is useful to consider also the ring $\mathcal{R}$ consisting of the union of the rings of rigid analytic functions over $K$ on annuli of the form $* \leq |t| < 1$ (commonly called the Robba ring over $K$). Note that $\Gamma_c$ is the subring of $\mathcal{R}$ consisting of Laurent series with coefficients in $W(k)$. Let $\mathcal{R}^+$ be the subring of $\mathcal{R}$ consisting of formal power series (i.e., with only nonnegative powers of $t$); this is the ring of rigid analytic functions on the open unit $t$-dist over $K$.

Define the categories

$$
\mathbf{F-Isoc}^\dagger(k[t]), \mathbf{F-Isoc}^\dagger(k((t)))
$$

to consist of finite projective modules over the respective rings $\mathcal{R}^+, \mathcal{R}$ equipped with compatible actions of $\sigma$ and $\frac{d}{dt}$ (note that this use of $\dagger$ is not standard notation). We then have faithful functors

$$
\mathbf{F-Isoc}(k[t]) \longrightarrow \mathbf{F-Isoc}^\dagger(k((t))) \longrightarrow \mathbf{F-Isoc}(k((t)))
$$

but no comparison between $\mathbf{F-Isoc}(k((t)))$ and $\mathbf{F-Isoc}^\dagger(k((t)))$.

Remark 2.11. One can also define convergent and overconvergent isocrystals without Frobenius structure (in both the global and local settings); on these larger categories, the fiber functors described in Remark 2.9 become neutral. This corresponds on the $\ell$-adic side to
passing from representations of arithmetic fundamental groups to representations of geometric fundamental groups. However, there are some subtleties hidden in the construction: one must include an additional condition on the convergence of the formal Taylor isomorphism (which is forced by the existence of a Frobenius structure).

3. Slopes

We next discuss a basic feature of isocrystals admitting no ℓ-adic analogue: the theory of slopes. We begin with the situation at a point.

**Definition 3.1.** Let $r, s$ be integers with $s > 0$ and $\gcd(r, s) = 1$. Let $\mathcal{F}_{r/s} \in \mathbf{F}_{\text{Isoc}}(k)$ be the object corresponding (via Remark 2.9) to the $K$-vector space on the basis $e_1, \ldots, e_s$ equipped with the $\phi$-action

$$\phi(e_1) = e_2, \ldots, \phi(e_{s-1}) = e_s, \quad \phi(e_s) = p^r e_1.$$

One checks easily that

$$\text{Hom}_{\mathbf{F}_{\text{Isoc}}(k)}(\mathcal{F}_{r/s}, \mathcal{F}_{r'/s'}) = \begin{cases} D_{r,s} & r'/s' = r/s \\ 0 & r'/s' \neq r/s \end{cases}$$

where $D_{r,s}$ denotes the division algebra over $K$ of degree $s$ and invariant $r/s$.

**Theorem 3.2** (Dieudonné–Manin). Suppose that $k$ is algebraically closed. Then every $\mathcal{E} \in \mathbf{F}_{\text{Isoc}}(\text{Spec } k)$ is uniquely isomorphic to a direct sum

$$\bigoplus_{r/s \in \mathbb{Q}} \mathcal{E}_{r/s}$$

in which each factor $\mathcal{E}_{r/s}$ is (not uniquely) isomorphic to a direct sum of copies of $\mathcal{F}_{r/s}$. (Note that uniqueness is forced by (3.1.1).

**Proof.** This is the standard Dieudonné-Manin classification theorem, the original reference for which is [64]. See also [54, Theorem 14.6.3] and [22].

**Definition 3.3.** For $\mathcal{E} \in \mathbf{F}_{\text{Isoc}}(k)$, choose an algebraically closed overfield $\overline{k}$ of $k$ and let $\mathcal{E}'$ be the pullback of $\mathcal{E}$ to $\mathbf{F}_{\text{Isoc}}(\overline{k})$. Then the direct sum decomposition of $\mathcal{E}$ given by Theorem 3.2 descends to $\mathcal{E}$ (and is independent of the choice of $\overline{k}$). We define the slope multiset of $\mathcal{E}$ to be the multisubset of $\mathbb{Q}$, of cardinality equal to the rank of $\mathcal{E}$, in which the multiplicity of $r/s$ equals rank $\mathcal{E}_{r/s}$; the slope multiset is additive in short exact sequences [33, Lemma 1.3.4]. We arrange the elements of the slope multiset into a convex Newton polygon with left endpoint $(0,0)$, called the slope polygon of $\mathcal{E}$. Note that the vertices of the slope polygon belong to $\{0, \text{rank}(\mathcal{E})\} \times \mathbb{Z}$.

For $\mathcal{E} \in \mathbf{F}_{\text{Isoc}}(X)$, we define the slope multiset and slope polygon of $\mathcal{E}$ at $x \in X$ by pullback to $\text{Spec } k(x)^{\text{perf}}$. We say that $\mathcal{E}$ is isoclinic if the slope multisets at all points are equal to a single repeated value; if that value is 0, we also say that $\mathcal{E}$ is unit-root or étale. By (3.1.1), there are no nonzero morphisms between isoclinic objects of distinct slopes.

**Remark 3.4.** Since the action of Frobenius on an object of $\mathbf{F}_{\text{Isoc}}(\text{Spec } k)$ can be characterized by writing down the matrix of action on a single basis, one might wonder whether the Newton polygon of the characteristic polynomial of said matrix coincides with the slope polygon. In general this is false; see [33, §1.3] for a counterexample. However, it does hold when the basis is the one derived from a cyclic vector for the action of Frobenius [43, Lemma 5.2.4].
Remark 3.5. Every \( E \in \mathbf{F-Isoc}(X) \) of rank 1 is isoclinic of some integer slope; this can either be proved directly or deduced from Theorem 3.12 below.

Remark 3.6. The sign convention for slopes used here is the one from [33]. However, in certain related contexts it is more natural to use the opposite sign convention. For example, in the theory of \( \varphi \)-modules over the Robba ring, the sign convention taken here is used in [39]; however, this theory can be reformulated in terms of vector bundles on curves \([29, 30, 31]\) and the opposite sign convention is the one consistent with geometric invariant theory.

Using slopes, we can now articulate two results that explain the relationship between \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local systems and isocrystals. The first result says that in a sense, there are “too few” \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local systems for them to serve as a good category of coefficient objects.

**Theorem 3.7** (Katz, Crew). The category of unit-root objects in \( \mathbf{F-Isoc}(X) \) is equivalent to the category of \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local systems on \( X \). In particular, if \( X \) is connected, this category is equivalent to the category of continuous representations of \( \pi_1(X, \overline{x}) \) on finite-dimensional \( \mathbb{Q}_p \)-vector spaces (for any geometric point \( \overline{x} \) of \( X \)).

*Proof.* See [13, Theorem 2.1]. \( \Box \)

The second result says that on the other hand, there are also “too many” \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local systems for them to serve as a good category of coefficient objects.

**Definition 3.8.** An \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local system \( V \) on \( X \) is unramified if the corresponding representations of the \( \acute{\text{e}} \text{tale} \) fundamental groups of the connected components of \( X \) restrict trivially to all inertia groups. If \( X \) admits an open immersion into a smooth proper variety \( \overline{X} \), then by Zariski-Nagata purity, \( V \) is unramified if and only if \( V \) extends (necessarily uniquely) to an \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local system on \( \overline{X} \). We say \( V \) is potentially unramified if there exists a finite \( \acute{\text{e}} \text{tale} \) cover \( X' \to X \) such that the pullback of \( V \) to \( X' \) is unramified.

**Theorem 3.9** (Tsuzuki). In the equivalence of Theorem 3.7, the unit-root objects in \( \mathbf{F-Isoc}^\dagger(X) \) form a full subcategory of \( \mathbf{F-Isoc}(X) \) corresponding to the category of potentially unramified \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local systems on \( X \).

*Proof.* In the case \( \dim X = 1 \), this is [74, Theorem 4.2.6]. For the general case, see [75, Theorem 1.3.1, Remark 7.3.1]. \( \Box \)

**Remark 3.10.** The local model of Theorem 3.7 is that the category of unit-root objects in \( \mathbf{F-Isoc}(k((t))) \) is equivalent to the category of continuous representations of the absolute Galois group \( G_{k((t))} \) on finite-dimensional \( \mathbb{Q}_p \)-vector spaces. The local model of Theorem 3.9 is that the unit-root objects in \( \mathbf{F-Isoc}^\dagger(k((t))) \) constitute the full subcategory in \( \mathbf{F-Isoc}(k((t))) \) corresponding to the representations with finite image of inertia. See [74, Theorem 4.2.6] for discussion of both statements.

**Remark 3.11.** By arguing as in [75], one may prove a common generalization of Theorem 3.7 and Theorem 3.9: for \( X \to Y \) an open immersion, the unit-root objects in \( \mathbf{F-Isoc}(X, Y) \) form a full subcategory corresponding to the category of \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local systems \( V \) on \( X \) with the following property: there exists some proper morphism \( Y' \to Y \) such that \( X' = X \times_Y Y' \) is finite \( \acute{\text{e}} \text{tale} \) over \( X \) and the pullback of \( V \) to \( X' \) extends to an \( \acute{\text{e}} \text{tale} \) \( \mathbb{Q}_p \)-local system on \( Y' \).

We now consider the variation of the slope polygon over \( X \).
Theorem 3.12 (Grothendieck, Katz, de Jong–Oort). For $E \in F\text{-Isoc}(X)$, the following statements hold.

(a) The slope polygon of $E$ is an upper semicontinuous function of $X$; moreover, its right endpoint is locally constant.

(b) The locus of points where the slope polygon does not coincide with its generic value (which by (a) is Zariski closed) is of pure codimension 1 in $X$.

Proof. For part (a), see [33, Theorem 2.3.1]. For part (b), see [19, Theorem 4.1] or Remark 5.4 below.

Remark 3.13. The reference given for Theorem 3.12(a) also implies the local model statement: for $E \in F\text{-Isoc}(k((t)))$, the slope polygon of the pullback of $E$ to $F\text{-Isoc}(k)$ (the special slope polygon) lies on or above the slope polygon of the pullback of $E$ to $F\text{-Isoc}(k((t)))$ (the generic slope polygon), with the same endpoint. This statement can be generalized to $E \in F\text{-Isoc}^\dagger(k((t)))$ using slope filtrations in $F\text{-Isoc}^\dagger(k((t)))$; see Remark 4.9.

4. Slope filtrations

We continue the discussion of slopes by considering filtrations by slopes. Such a filtration is loosely analogous to the filtration occurring in the definition of a variation of Hodge structures.

Theorem 4.1 (Katz). Suppose $E \in F\text{-Isoc}(X)$ has the property that the point $(m,n) \in \mathbb{Z}^2$ is a vertex of the slope polygon at every point of $E$. Then there exists a short exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

in $F\text{-Isoc}(X)$ with rank $E_1 = m$ such that for each $x \in X$, the slope polygon of $E_1$ is the portion of the slope polygon of $E$ from $(0,0)$ to $(m,n)$.

Proof. This is a straightforward consequence of [33, Theorem 2.4.2].

Corollary 4.2 (Katz). Suppose $E \in F\text{-Isoc}(X)$ has the property that the slope polygon of $E$ is constant on $X$. Then $E$ admits a unique filtration

$$0 = E_0 \subset \cdots \subset E_l = E$$

such that each successive quotient $E_i/E_{i-1}$ is everywhere isoclinic of some slope $s_i$, and $s_1 < \cdots < s_l$. We call this the slope filtration of $E$.

Remark 4.3. One local model of Corollary 4.2 is that every object of $F\text{-Isoc}(k((t)))$ has a slope filtration [39, Proposition 5.10]. A more substantial version is that for $E \in F\text{-Isoc}(k[[t]])$, if the generic and special slope polygons coincide, then $E$ admits a slope filtration [33, Corollary 2.6.3]. A similar statement holds for $E \in F\text{-Isoc}^\dagger(k((t)))$; see Remark 4.9.

Remark 4.4. The arguments in [33] involve a finite projective module equipped only with a Frobenius action (and not an integrable connection). On one hand, this means that Theorem 4.1 remains valid in this setting, as does its local model (Remark 4.3). On the other hand, to obtain Theorem 4.1 (or Remark 4.3) as stated, one must make an extra argument to verify that the filtration is respected also by the connection. To wit, the Kodaira-Spencer construction defines a morphism $E_1 \to E_2$ of $\sigma$-modules which vanishes if and only if $E_1$ is stable under the connection; however, this vanishing is provided by (3.1.1).
There is no analogue of Theorem 4.1 for overconvergent $F$-isocrystals. Here is an explicit example.

**Example 4.5.** Let $X$ be the modular curve $X(N)$ for some $N \geq 3$ not divisible by $p$ (taking $N \geq 3$ forces this to be a scheme rather than a Deligne-Mumford stack). Then the crystalline $H^1$ of the universal elliptic curve over $X$ is an object $E$ of $F$-$\text{Isoc}^1(X)$ of rank 2. The slope polygon of $E$ generically has slopes 0, 1, but there is a finite set $Z \subset X$ (the *supersingular locus*) at which the slope polygon jumps to 1/2, 1/2. Let $U$ be the complement of $Z$ in $X$ (the *ordinary locus*); by Theorem 4.1, the restriction of $E$ to $F$-$\text{Isoc}(U)$ admits a rank 1 subobject which is unit-root. However, no such subobject exists in $F$-$\text{Isoc}^1(U)$; see Remark 5.12.

By completing at a supersingular point, we also obtain an irreducible object of $F$-$\text{Isoc}(k[[t]])$ which remains irreducible in $F$-$\text{Isoc}^1(k((t)))$ but not in $F$-$\text{Isoc}(k((t)))$.

**Remark 4.6.** Notwithstanding Example 4.5, one can formulate something like a filtration theorem for overconvergent $F$-isocrystals, at the expense of working in a “perfect” setting (i.e., where the Frobenius lift is a bijection); since one cannot differentiate in such a setting, one only gets statements about individual liftings.

For simplicity, we discuss only the local model situation here. Put $\Gamma_{\text{perf}} = W(k((t)))$; there is a natural Frobenius-equivariant embedding $\Gamma \to \Gamma_{\text{perf}}$ taking $t$ to the Teichmüller lift $[t]$ (that is, the Frobenius lift $\sigma$ on $\Gamma$ corresponds to the unique Frobenius lift $\varphi$ on $\Gamma_{\text{perf}}$). Each element of $\Gamma_{\text{perf}}$ can be written uniquely as a $p$-adically convergent series $\sum_{n=0}^{\infty} p^n [\pi_n]$ for some $\pi_n \in k((t))_{\text{perf}}$, let $\Gamma_{\text{perf}}^\infty$ be the subset of $\Gamma_{\text{perf}}$ consisting of those series for which the $t$-adic valuations of $\pi_n$ are bounded below by some linear function of $n$ (for $n > 0$). One verifies easily that $\Gamma_{\text{perf}}^\infty$ is a $\varphi$-stable subring of $\Gamma_{\text{perf}}$ containing the image of $\Gamma_c$.

Suppose now that $E$ is a finite projective module over $\Gamma_{\text{perf}}[p^{-1}]$ equipped with an isomorphism $\varphi^*E \cong E$. Using an argument of de Jong [17, Proposition 5.5], one can show [39, Proposition 5.11] that $E$ admits a unique filtration implies that $E$ admits a unique filtration

$$0 = E_0 \subset \cdots \subset E_l = E$$

by $\varphi$-stable submodules such that each successive quotient $E_i/E_{i-1}$ is everywhere isoclinic of some slope $s_i$, and $s_1 > \cdots > s_l$. We call this the *reverse slope filtration* of $E$.

We add some additional remarks concerning the local situation.

**Remark 4.7.** For $E \in F$-$\text{Isoc}^1(k[[t]])$, an argument of Dwork [17, Lemma 6.3] implies that $E$ admits a unique filtration specializing to the slope filtration in $F$-$\text{Isoc}(k)$, and that each subquotient descends uniquely to an isoclinic object in $F$-$\text{Isoc}(k[[t]])$. In particular, the image of $E$ in $F$-$\text{Isoc}^1(k((t)))$ admits a filtration that in a certain sense reflects the special slope polygon of $E$. This sense is made more precise in Remark 4.9 below.

**Remark 4.8.** The functor from $F$-$\text{Isoc}^1(k((t)))$ to $F$-$\text{Isoc}^1(k((t)))$ is not fully faithful in general, but it is fully faithful on the category of isoclinic objects of any fixed slope [43, Theorem 6.3.3(b)]. We declare an object of $F$-$\text{Isoc}^1(k((t)))$ to be *isoclinic* of a particular slope if it arises from an isoclinic object of $F$-$\text{Isoc}^1(k((t)))$ of that slope.

Beware that the analogue of (3.1.1) in this context only holds when $r/s \leq r'/s'$. More precisely, if $E_1, E_2 \in F$-$\text{Isoc}^1(k((t)))$ are isoclinic of slopes $s_1, s_2$, then $\text{Hom}_{F$-$\text{Isoc}^1(k((t)))}(E_1, E_2)$ vanishes when $s_1 < s_2$ (by [43, Proposition 3.3.4]), equals the corresponding Hom-set in $F$-$\text{Isoc}^1(k((t)))$ if $s_1 = s_2$ (by the full faithfulness statement quoted above), and is hard to control if $s_1 > s_2$. 

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Remark 4.9. In light of Remark 4.9, one may ask whether an arbitrary object $E \in F\text{-Isoc}^\dagger(k((t)))$ admits a slope filtration in the sense of Corollary 4.2. Such a filtration, were it to exist, would be unique by virtue of Remark 4.8; namely, under the geometric sign convention (Remark 3.6), it would coincide with the Harder-Narasimhan filtration by destabilizing subobjects. However, constructing such a filtration is made difficult by the fact that in this setting, it cannot be studied using cyclic vectors (as in Remark 3.4). Nonetheless, with some effort one can prove existence of such a filtration [39, Theorem 6.10] (again using the Kodaira-Spencer argument to pass from a filtration of $\sigma$-modules to a filtration of isocrystals) and then use it to define the slope polygon of $E$. (For alternate expositions of the construction, see [43, Theorem 6.4.1], [49, Theorem 1.7.1].)

For $E \in F\text{-Isoc}^\dagger(k((t)))$, one can now associate two slope polygons to $E$: one arising from the image in $F\text{-Isoc}(k((t)))$, called the generic slope polygon; and one arising from the image in $F\text{-Isoc}^\dagger(k((t)))$, called the special slope polygon. In case $E$ arises from $F\text{-Isoc}(k[[t]])$, these definitions agree with the ones from Remark 3.13. One can make an extended Robba ring containing both $\Gamma_c^\text{perf}$ and $R$ and use it to compare the slope filtration described above with the reverse slope filtration (Remark 4.6), so as to obtain analogues of Remark 3.13 and Remark 4.3: the special slope polygon again lies on or above the generic slope polygon, with the same right endpoint [43, Proposition 5.5.1], and equality implies the existence of a slope filtration of $E$ itself [43, Theorem 5.5.2].

Remark 4.10. By combining Remark 3.10 with Remark 4.9, one sees that every object $E \in F\text{-Isoc}^\dagger(k((t)))$ admits a filtration by objects such that for some finite separable morphism $\text{Spec} k'(((u))) \to \text{Spec} k((t))$, the pullback to $F\text{-Isoc}^\dagger(k'((u)))$ of each subquotient of the filtration is itself an object arising by pullback from pullback from $F\text{-Isoc}(k')$. (Technical note: forming the pullback involves changing Frobenius lifts, which is achieved using the Taylor isomorphism provided by the connection.) This is a statement formulated (although not formally conjectured) by Crew [15, §10.1], commonly known thereafter as Crew’s conjecture; the approach to Crew’s conjecture we have just described is the one given in [39]. Independent contemporaneous proofs were given by André [5] and Mebkhout [65] based on the theory of $p$-adic differential equations; see [54, Theorem 20.1.4] for a similar argument.

Remark 4.11. Let $X$ be a curve, let $x \in X$ be a closed point of residue field $k$, let $U$ be the complement of $x$ in $U$, and identify the completed local ring of $X$ at $x$ with $k[[t]]$. For $E \in F\text{-Isoc}(U,X)$, by applying Remark 4.10 to the pullback of $E$ to $F\text{-Isoc}^\dagger(k((t)))$, we obtain a representation of $G_{k((t))}$ with finite image of inertia. This is called the local monodromy representation of $E$ at $x$, because it plays a similar role to that played in $\ell$-adic cohomology to the pullback of a local system from $X$ to $\text{Spec} k((t))$; see Remark 7.7 for more details. For this reason, Crew’s conjecture is also called the $p$-adic local monodromy theorem; however, in the $p$-adic setting there is no natural definition of a global monodromy representation which specializes to the local ones.

5. Restriction functors

Throughout §5, let $X \to Y$ be an open immersion of $k$-varieties, let $U$ be an open dense subscheme of $X$, and let $W$ be an open subscheme of $Y$ containing $U$. We exhibit some properties of the restriction functor $F\text{-Isoc}(X,Y) \to F\text{-Isoc}(U,W)$; in the case of unit-root isocrystals, most of these statements can be predicted from Theorem 3.7 and Theorem 3.9,
but the proofs require additional ideas. In a few cases, the predictions turn out to be misleading.

We begin with an analogue of Zariski-Nagata purity (which has no local model). In the unit-root case, this may be deduced from Remark 3.11.

**Theorem 5.1** (Kedlaya, Shiho). *Suppose that $\text{codim}(X - U, X) \geq 2$ and $\text{codim}(Y - W, Y) \geq 2$. Then the functor $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(U, W)$ is an equivalence of categories. In particular, the functors $\mathbf{F-Isoc}(X) \to \mathbf{F-Isoc}(U)$, $\mathbf{F-Isoc}^\dagger(X) \to \mathbf{F-Isoc}^\dagger(U)$, $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(U, Y)$ are equivalences of categories.*

*Proof.* See [72, Theorem 3.1]. For the claim about $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(U, Y)$ (and by extension $\mathbf{F-Isoc}^\dagger(X) \to \mathbf{F-Isoc}^\dagger(U)$), see also [46, Proposition 5.3.3]. □

**Corollary 5.2** (Kedlaya). *The functors $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(U, Y) \times_{\mathbf{F-Isoc}(U)} \mathbf{F-Isoc}(X)$, $\mathbf{F-Isoc}^\dagger(X) \to \mathbf{F-Isoc}^\dagger(U) \times_{\mathbf{F-Isoc}(U)} \mathbf{F-Isoc}(X)$ are equivalences of categories.*

*Proof.* Using Theorem 5.1, one may reduce the first equivalence to the case where $Y - U$ is a disjoint union of irreducible divisors and $X - U$ is the union of some subset among these. We may then work locally on $Y$ to reduce to the case where $Y - U$ consists of one irreducible divisor, in which case either $X = U$ or $X = Y$ and the claim is straightforward. With the first equivalence in hand, one immediately deduces the second. Alternatively, see [46, Proposition 5.3.7]. □

**Remark 5.3.** In the case where $\dim(X) = 1$, Corollary 5.2 admits a local variant: if $Y - X$ consists of a single $k$-rational point $x$, for $t$ a uniformizer of $Y$ at $x$, the functor $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(X) \times_{\mathbf{F-Isoc}(Y)} \mathbf{F-Isoc}(k[[t]])$ is an equivalence.

**Remark 5.4.** Using Theorem 5.1, we may recover the de Jong–Oort theorem on purity of the Newton polygon stratification (Theorem 3.12(b)) as follows. We may assume that $X$ is irreducible with generic point $\eta$. For $E \in \mathbf{F-Isoc}(X)$, let $U$ be the subset of $X$ at which the slope polygon coincides with its value at $\eta$. By Theorem 3.12(a), $U$ is open; by Corollary 4.2, the restriction of $E$ to $U$ admits a slope filtration. If $\text{codim}(X - U, X) \geq 2$, then by Theorem 5.1 this filtration extends over $X$; this proves the claim.

We continue with a general statement about restriction functors, which combines work of several authors; in addition to the results cited in the proof, see Remark 5.7 and Remark 5.8 for relevant attributions.

**Theorem 5.5** (de Jong, Drinfeld–Kedlaya, Kedlaya, Shiho). *The restriction functor $\mathbf{F-Isoc}(X, Y) \to \mathbf{F-Isoc}(U, W)$...*
is fully faithful. In particular, the functors
\[
F\text{-Isoc}(X, Y) \to F\text{-Isoc}(X), \quad F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}(X),
\]
\[
F\text{-Isoc}(X, Y) \to F\text{-Isoc}(U, Y), \quad F\text{-Isoc}(X) \to F\text{-Isoc}(U), \quad F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}^\dagger(U)
\]
are fully faithful.

Proof. By forming the composition
\[
F\text{-Isoc}(X, Y) \to F\text{-Isoc}(U, W) \to F\text{-Isoc}(U),
\]
we immediately reduce the general problem to the case \(W = U\). In this case, the functor in question factors as
\[
F\text{-Isoc}(X, Y) \to F\text{-Isoc}(X) = F\text{-Isoc}(X, X) \to F\text{-Isoc}(U, X) \to F\text{-Isoc}(U).
\]
By [46, Theorem 5.2.1], the functor \(F\text{-Isoc}(X, X) \to F\text{-Isoc}(U, X)\) is fully faithful. By [24, Theorem 2.3.3], the functors \(F\text{-Isoc}(X, Y) \to F\text{-Isoc}(X), \ F\text{-Isoc}(U, X) \to F\text{-Isoc}(U)\) are fully faithful (see Lemma A.1 for an alternate proof).

Corollary 5.6. Let \(f : Y' \to Y\) be a finite flat morphism such that \(X' = f^{-1}(X)\) is finite étale and surjective over \(X\). Then \(\mathcal{E} \in F\text{-Isoc}(X)\) extends to \(F\text{-Isoc}(X, Y')\) if and only if \(f^*\mathcal{E} \in F\text{-Isoc}(X')\) extends to \(F\text{-Isoc}(X', Y')\). In particular, \(\mathcal{E} \in F\text{-Isoc}(X)\) extends to \(F\text{-Isoc}^\dagger(X, Y')\) if and only if \(f^*\mathcal{E} \in F\text{-Isoc}^\dagger(X')\).

Proof. If \(f^*\mathcal{E} \in F\text{-Isoc}(X')\) extends to \(\mathcal{F} \in F\text{-Isoc}(X', Y')\), then using Remark 2.7, the restriction of \(f_*\mathcal{F} \in F\text{-Isoc}(X, Y)\) to \(F\text{-Isoc}(X)\) has a summand isomorphic to \(\mathcal{E}\). By Theorem 5.5, the decomposition extends to a decomposition of \(f_*\mathcal{F}\) itself.

Remark 5.7. For unit-root isocrystals, the full faithfulness of \(F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}(X)\) is included in Theorem 3.9; the general case is treated in [40, Theorem 1.1]. The proofs of full faithfulness of \(F\text{-Isoc}(X, Y) \to F\text{-Isoc}(X)\) appearing in [24, Theorem 2.3.3] and in Lemma A.1 are small variants of the proof of [40, Theorem 1.1].

The full faithfulness of \(F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}^\dagger(U)\) follows from [27, Théorème 4]. The argument is extended in [46, Theorem 5.2.1] to obtain full faithfulness of \(F\text{-Isoc}(X, Y) \to F\text{-Isoc}(U, Y)\); see also [70] for some stronger results.

Remark 5.8. The local model of Theorem 5.5 is the statement that the functors
\[
F\text{-Isoc}(k[t]) \to F\text{-Isoc}^\dagger(k((t))), \ F\text{-Isoc}^\dagger(k((t))) \to F\text{-Isoc}(k((t)))
\]
are fully faithful. The full faithfulness of the composite functor \(F\text{-Isoc}(k[t]) \to F\text{-Isoc}(k((t)))\) is due to de Jong [17, Theorem 9.1], and is the key ingredient in his proof of the analogue of Tate’s extension theorem for \(p\)-divisible groups in equal positive characteristic. (See also [42, Theorem 1.1] for a streamlined exposition.)

In fact, de Jong’s approach is to first show that \(F\text{-Isoc}(k[t]) \to F\text{-Isoc}^\dagger(k((t)))\) is fully faithful, then to show that the restriction of \(F\text{-Isoc}^\dagger(k((t))) \to F\text{-Isoc}(k((t)))\) to the essential image of \(F\text{-Isoc}(k[t])\) is fully faithful. Both steps make essential use of the functor \(F\text{-Isoc}^\dagger(k((t))) \to F\text{-Isoc}^\dagger(k[t])\); for example, it is crucial that objects of \(F\text{-Isoc}(k[t])\) admit slope filtrations in \(F\text{-Isoc}^\dagger(k[t])\) (Remark 4.7). The argument also makes essential use of the reverse slope filtration (Remark 4.6).
Building on de Jong’s approach full faithfulness of $F\text{-Isoc}^\dagger(k((t))) \to F\text{-Isoc}(k((t)))$ was established in [40, Theorem 5.1]. The argument follows [17] fairly closely, except that Remark 4.7 is replaced by Remark 4.9 (see also Remark 7.7). Using this local statement, one then obtains full faithfulness of $F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}(X)$ [40, Theorem 1.1]; the argument is similar to that given below in the proof of Theorem 5.5.

Although this is not explained in [17], one may use the results of that paper to establish full faithfulness of $F\text{-Isoc}(X) \to F\text{-Isoc}(U)$. However, even if one does this, the argument still implicitly refers to $F\text{-Isoc}^\dagger(X)$; in fact, despite the fact that the statement can be formulated using only convergent isocrystals, we know of no proof that entirely avoids the use of overconvergent isocrystals.

**Remark 5.9.** If one considers isocrystals without Frobenius structure, then the analogue of full faithfulness for $F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}^\dagger(U)$ holds (by the same references as in Remark 5.7). But the analogue of full faithfulness for $F\text{-Isoc}^\dagger(X)$ to $F\text{-Isoc}(X)$ fails (see [1]). The latter is related to known pathologies in the theory of $p$-adic differential equations related to $p$-adic Liouville numbers (i.e., $p$-adic integers which are overly well approximated by ordinary integers); see [57] for more discussion.

**Remark 5.10.** An alternate approach to the full faithfulness problem for $F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}(X)$, which does not go through the local model and does not require Crew’s conjecture, is suggested by recent work of Ertl [26] on an analogous problem in de Rham-Witt cohomology.

We next consider extension of subobjects.

**Theorem 5.11** (Kedlaya). *Any subobject in $F\text{-Isoc}(U,Y)$ of an object of $F\text{-Isoc}(X,Y)$ extends to $F\text{-Isoc}(X,Y)$. In particular, any subobject in $F\text{-Isoc}^\dagger(U)$ of an object of $F\text{-Isoc}^\dagger(X)$ extends to $F\text{-Isoc}^\dagger(X)$.*

**Proof.** See [46, Proposition 5.3.1].

**Remark 5.12.** By contrast with Theorem 5.11, not every subobject in $F\text{-Isoc}(X)$ of an object of $F\text{-Isoc}^\dagger(X)$ extends to $F\text{-Isoc}^\dagger(X)$. For example, set notation as in Example 4.5. If the unit-root subobject of $\mathcal{E}$ in $F\text{-Isoc}(U)$ could be extended to $F\text{-Isoc}^\dagger(U)$, then by Theorem 5.5 and Corollary 5.2 it would also extend to $F\text{-Isoc}^\dagger(X)$; this would imply that for any point $x \in X$ in the supersingular locus, the rigid cohomology of the elliptic curve $E_x$ corresponding to $x$ contains a distinguished line. However, using the endomorphism ring of such a curve (which is an order in a quaternion algebra over $\mathbb{Q}$) one sees easily that no such distinguished line can exist.

**Remark 5.13.** Given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

with $\mathcal{E}_1, \mathcal{E}_2 \in F\text{-Isoc}^\dagger(X)$ and $\mathcal{E} \in F\text{-Isoc}(X)$, it does not follow that $\mathcal{E} \in F\text{-Isoc}^\dagger(X)$; for instance, this already fails in case $X = \mathbb{A}^1_k$ and $\mathcal{E}_1, \mathcal{E}_2$ are both the constant object in $F\text{-Isoc}^\dagger(X)$. Similarly, if $\mathcal{E}_1, \mathcal{E}_2 \in F\text{-Isoc}^\dagger(X)$ and $\mathcal{E} \in F\text{-Isoc}^\dagger(U)$, it does not follow that $\mathcal{E} \in F\text{-Isoc}^\dagger(X)$ unless we allow for log structures (see Definition 7.1).

One can ask whether extendability of an $F$-isocrystal can be characterized on the level of curves (note that this question has no local model). Here is an example of such a statement.
Theorem 5.14 (Shiho). The following statements hold.

(a) An object of $\text{F-Isoc}(U, Y)$ extends to $\text{F-Isoc}(X, Y)$ if and only if for every curve $C \subseteq Y$, the pullback object in $\text{F-Isoc}(C \times_Y U, C)$ extends to $\text{F-Isoc}(C \times_Y X, C)$.

(b) An object of $\text{F-Isoc}^1(U)$ lifts to $\text{F-Isoc}^1(X)$ if and only if for every curve $C \subseteq X$, the pullback object in $\text{F-Isoc}^1(C \times_X U)$ lifts to $\text{F-Isoc}^1(C)$.

Proof. We obtain (a) by applying [73, Theorem 0.1]. This immediately implies (b). □

In a previous version of this notes, the following statements were mistakenly attributed to Shiho [71]; in fact, the main result of loc. cit. is somewhat weaker (for instance, one must assume that the underlying connection extends to a strict neighborhood), so we leave this as a conjecture.

Conjecture 5.15. An object of $\text{F-Isoc}(X)$ extends to $\text{F-Isoc}(X, Y)$ if and only if for every curve $C \subseteq Y$, the pullback object in $\text{F-Isoc}(C \times_Y X)$ extends to $\text{F-Isoc}(C \times_Y X, C)$. In particular, an object of $\text{F-Isoc}(X)$ extends to $\text{F-Isoc}^1(X)$ if and only if for every curve $C \subseteq X$, the pullback object in $\text{F-Isoc}(C)$ extends to $\text{F-Isoc}^1(C)$. (This holds for unit-root objects by Theorem 3.7 and Theorem 3.9.)

Remark 5.16. In conjunction with Theorem 5.14, Conjecture 5.15 would imply that an object of $\text{F-Isoc}(U)$ extends to $\text{F-Isoc}(X)$ if and only if for every curve $C \subseteq X$, the pullback object in $\text{F-Isoc}(C \times_Y U)$ extends to $\text{F-Isoc}(C)$. (Again, this holds for unit-root objects by Theorem 3.7.)

One expects the following by analogy with Wiesend’s theorem in the $\ell$-adic case [76, 23], but we have no approach in mind.

Conjecture 5.17. For $\mathcal{E} \in \text{F-Isoc}^1(X)$ irreducible, we can find a curve $C \subseteq X$ such that the pullback of $\mathcal{E}$ to $\text{F-Isoc}^1(C)$ is irreducible.

Remark 5.18. In light of Remark 5.12, Conjecture 5.17 cannot be proved by reduction from $\text{F-Isoc}^1(X)$ to $\text{F-Isoc}(X)$.

6. Slope gaps

We next study the behavior of gaps between slopes, starting with a cautionary remark.

Remark 6.1. Note that in general, a persistent gap between slopes is not enough to guarantee the existence of a slope filtration. That is, suppose that $\mathcal{E} \in \text{F-Isoc}(X)$ has the property that for some positive integer $k < \text{rank}(\mathcal{E})$, the $k$-th and $(k + 1)$-st smallest slopes of $\mathcal{E}$ at each point of $X$ are distinct. Then $\mathcal{E}$ need not admit a subobject of rank $k$ whose slopes at each point are precisely the $k$ smallest slopes of $\mathcal{E}$ at that point. Namely, by Theorem 3.12, this would imply that the sum of the $k$ smallest slopes is locally constant, which can fail in examples (see Example 6.2). However, this does hold if the gap is large enough; see Theorem 6.3.

Example 6.2. Let $Y$ be the moduli space of principally polarized abelian threefolds with full level $N$ structure for some $N \geq 3$ not divisible by $p$. Then the crystalline $H^1$ of the universal abelian variety over $Y$ is an object $\mathcal{E}$ of $\text{F-Isoc}^1(Y)$ of rank 6. It is known (e.g., see [12]) that the image of the slope polygon map for $\mathcal{E}$ consists of all Newton polygons with nonnegative slopes and right endpoint $(6, 3)$. In particular, we can find a curve $X$ in $Y$ such
that the pullback of $E$ to $X$ has slopes $0, 0, 0, 1, 1, 1$ at its generic point and $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}$ at some closed point. Since the smallest $3$ slopes do not have constant sum, they cannot be isolated using a slope filtration.

Recall that there is a loose analogy between isocrystals and variations of Hodge structure. With Griffiths transversality in mind, one may ask whether a persistent gap between slopes of length greater than $1$ gives rise to a partial slope filtration. In fact, an even stronger statement holds: it is enough for such a gap to occur generically.

**Theorem 6.3** (Drinfeld–Kedlaya). Suppose that $E \in F\text{-Isoc}(X)$ (resp. $E \in F\text{-Isoc}^\dagger(X)$) has the property that for some positive integer $k$, the difference between the $k$-th and $(k+1)$-st smallest slopes of $E$ at each generic point of $X$ is strictly greater than $1$.

(a) At each $x \in X$, the sum of the $k$ smallest slopes of $E_x$ is equal to a locally constant value, and the difference between the $k$-th and $(k+1)$-st smallest slopes of $E$ is strictly greater than $1$.

(b) There is a splitting $E \cong E_1 \oplus E_2$ of $E$ in $F\text{-Isoc}(X)$ (resp. $F\text{-Isoc}^\dagger(X)$) with $\text{rank}(E_1) = k$ such that the slopes of $E_1$ at each point are exactly the $k$ smallest slopes of $E$ at that point.

**Proof.** In light of Theorem 5.5, it is only necessary to prove Theorem 6.3 in the case $E \in F\text{-Isoc}(X)$. This is proved in [24, Theorem 1.1.4] using the Cartier operator; see Lemma A.3 for a variant proof. □

**Remark 6.4.** Theorem 6.3 implies that if $X$ is irreducible and $E \in F\text{-Isoc}^\dagger(X)$ is indecomposable, then there is no gap of length greater than $1$ between consecutive slopes of $E$ at the generic point of $X$. However, such gaps can occur at other points of $X$; see [24, Appendix] for some examples.

**Remark 6.5.** Theorem 6.3 can be used to obtain nontrivial consequences about the Newton polygons of Weil $\mathbb{Q}_\ell$-sheaves, refining results of V. Lafforgue [60]. See [24] for more discussion.

### 7. Logarithmic compactifications

As in other cohomology theories, a key technical tool in the study of overconvergent $F$-isocrystals on nonproper varieties is the formation of certain logarithmic compactifications.

**Definition 7.1.** Suppose that $X \to \overline{X}$ is an open immersion with $\overline{X}$ smooth and $\overline{X} - X$ a normal crossings divisor. Equip $\overline{X}$ with the corresponding log structure; one can then define the associated category $F\text{-Isoc}(\overline{X})$ of convergent log-$F$-isocrystals.

To give a local description of this category, suppose that there exist a smooth affine formal scheme $P$ over $W(k)$ with $P_k \cong \overline{X}$, a relative normal crossings divisor $Z$ on $P$ with $Z_k \cong \overline{X} - X$, and a Frobenius lift $\sigma : P \to P$ which acts on $Z$. Then an object of $F\text{-Isoc}(\overline{X})$ may be viewed as a vector bundle $E$ on $P_K$ equipped with an integrable logarithmic connection and an isomorphism $\sigma^* E \to E$ of logarithmic $\mathcal{O}$-modules.

**Definition 7.2.** Given an integrable logarithmic connection, the resulting map $E \to E \otimes \Omega^\log_{P_K}/\Omega_{P_K}/K$ induces an $\mathcal{O}_{Z_K}$-linear endomorphism of $E|_{Z_K}$ called the residue map. The eigenvalues of the residue map must be killed by differentiation, and thus belong to $K$; the presence of the Frobenius structure forces the set of eigenvalues to be stable under multiplication by $p$. That is, any object of $F\text{-Isoc}(\overline{X})$ has nilpotent residue map. Note that
this would fail if we only required $\sigma^*\mathcal{E} \to \mathcal{E}$ to be an isomorphism away from $Z_K$; in this case, only the reductions modulo $\mathbb{Z}$ of the eigenvalues of the residue map would form a set stable under multiplication by $p$, so they would only be constrained to be rational numbers.

**Theorem 7.3** (Kedlaya). The functor $\text{F-Isoc}(\overline{X}) \rightarrow \text{F-Isoc}(X, \overline{X})$ is fully faithful. In particular, if $\overline{X}$ is proper, then $\text{F-Isoc}(\overline{X}) \rightarrow \text{F-Isoc}^1(X)$ is fully faithful.

*Proof.* See [46, Theorem 6.4.5].

Theorem 5.14 admits the following logarithmic analogue.

**Theorem 7.4** (Shiho). An object of $\text{F-Isoc}^1(X)$ extends to $\text{F-Isoc}(\overline{X})$ if and only if for every curve $C \subseteq \overline{X}$, the pullback object in $\text{F-Isoc}^1(C \times_{\overline{X}} X)$ extends to $\text{F-Isoc}(C)$.

*Proof.* Again, see [73, Theorem 0.1].

**Remark 7.5.** In light of Theorem 7.4, Conjecture 5.15 would imply that an object of $\text{F-Isoc}(X)$ extends to $\text{F-Isoc}(\overline{X})$ if and only if for every curve $C \subseteq \overline{X}$, the pullback object in $\text{F-Isoc}(C \times_{\overline{X}} X)$ extends to $\text{F-Isoc}(C)$.

In general, not every object of $\text{F-Isoc}^1(X)$ extends to $\text{F-Isoc}(\overline{X})$. However, the obstruction to extending can always be eliminated using a finite cover of varieties. Note that the unit-root case of the following theorem is an immediate consequence of Theorem 3.9.

**Theorem 7.6** (Kedlaya). Given $\mathcal{E} \in \text{F-Isoc}^1(X)$, there exist an alteration $f : X' \to X$ in the sense of de Jong [16] and an open immersion $j : X' \to \overline{X}'$ with $\overline{X}'$ smooth proper and $\overline{X}' - X'$ a normal crossings divisor, such that the pullback of $\mathcal{E}$ to $\text{F-Isoc}^1(X')$ extends to $\text{F-Isoc}(\overline{X}')$.

*Proof.* For the case dim $X = 1$, see [38, Theorem 1.1]. For the general case, see [55, Theorem 5.0.1].

**Remark 7.7.** The local model of Theorem 7.6 is the following statement: for any $\mathcal{E} \in \text{F-Isoc}^1(k((t)))$, there exists a finite separable morphism $\text{Spec } k'((u)) \to \text{Spec } k((t))$ such that the pullback of $\mathcal{E}$ to $\text{F-Isoc}^1(k'((u)))$ extends to the category $\text{F-Isoc}(k[[u]]_{\log})$ of finite projective $\Omega[p^{-1}]$-modules equipped with compatible actions of $\sigma$ and $u^d \frac{d}{du}$. This statement was conjectured by de Jong [18, §5]; it is now known as a corollary of Crew’s conjecture (see Remark 4.10). More precisely, $\mathcal{E} \in \text{F-Isoc}^1(k((t)))$ lifts to $\text{F-Isoc}(k[[t]]_{\log})$ if and only if its image in $\text{F-Isoc}^1(k((t)))$ is a successive extension of objects, each of which arises by pullback from $\text{F-Isoc}(k)$.

**Remark 7.8.** In the case dim $X = 1$, Theorem 7.6 is an easy consequence of the local model statement described in Remark 7.7. The general case, originally conjectured by Shiho [69, Conjecture 3.1.8], is much harder: it is the culmination of the sequence of papers [46, 47, 50, 55], where it is described as a semistable reduction theorem for overconvergent $F$-isocrystals. The difficulty in the higher-dimensional case is that the alteration is generally forced to include some wildly ramified cover, whose singularities are hard to control; consequently, one cannot simply argue using Theorem 5.1 and the one-dimensional case. Rather, one must work locally on the Riemann-Zariski space of the variety. Similar difficulties arise in trying to formulate a higher-dimensional analogue of the formal classification of meromorphic differential equations; see [52, 53].
Remark 7.9. Note that de Jong’s alteration theorem is required even to produce the pair $X', \overline{X}'$ with the prescribed smoothness properties; the nature of de Jong’s proof is such that one has very little control over the finite locus of the alteration. One might hope that under a strong hypothesis on resolution of singularities, Theorem 7.6 can be strengthened to ensure that the alteration $f$ is finite étale over $X$. This can be achieved when $\dim X = 1$: it is enough to ensure that $f$ trivializes the local monodromy representations (Remark 4.11), which can be achieved via careful use of Katz-Gabber local-to-global extensions [34]. However, it is less clear whether one should even expect this to be possible when $\dim X > 1$, as there is in general no global monodromy representation controlling the situation.

8. Cohomology

Having studied the coefficient objects of rigid cohomology up to now, it is finally time to introduce the cohomology theory itself. Again, we fall back on [62] for missing foundational discussion.

Definition 8.1. For $i \geq 0$ and $E \in \text{F-Isoc}^\dagger(X)$, let $H^i_{\text{rig}}(X, E)$ denote the $i$-th rigid cohomology group of $X$ with coefficients in $E$; it is a $K$-vector space equipped with an isomorphism with its $\varphi$-pullback.

One may describe rigid cohomology concretely in case $X$ is affine. Let $P$ be a smooth affine formal scheme with $P_k \cong X$; then $E$ can be realized as a vector bundle with integrable connection on a strict neighborhood $U$ of $P_K$ in a suitable ambient space. The rigid cohomology is then obtained by taking the hypercohomology of the de Rham complex

$$0 \to E \xrightarrow{\nabla} E \otimes_{O_U} \Omega^1_{U/K} \xrightarrow{\nabla} E \otimes_{O_U} \Omega^2_{U/K} \to \cdots,$$

then taking the direct limit over (decreasing) strict neighborhoods. For example, if $X = \mathbb{A}^n_k$, we may take $P$ to be the formal affine $n$-space, identify $P_K$ with the closed unit polydisc in $T_1, \ldots, T_n$, then take the family of strict neighborhoods to be polydiscs of radii strictly greater than 1.

Remark 8.2. For constant coefficients, the computation of rigid cohomology in the affine case agrees with the definition of “formal cohomology” by Monsky–Washnitzer [66], which was one of Berthelot’s motivations for the definition of rigid cohomology. The key example is that of the affine line with constant coefficients: the de Rham complex over the closed unit disc has infinite-dimensional cohomology, whereas rigid cohomology behaves as one would expect from the Poincaré lemma (i.e., $H^0$ is one-dimensional and $H^1$ vanishes).

Theorem 8.3 (Ogus). Suppose that $X$ is smooth and proper, and let $E$ be the object of $\text{F-Isoc}(X) = \text{F-Isoc}^\dagger(X)$ corresponding to a crystal $M$ of finite $O_{X, \text{crys}}$-modules via Theorem 2.2. Then there are canonical isomorphisms

$$H^i(X_{\text{crys}}, M) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^i_{\text{rig}}(X, E) \quad (i \geq 0).$$

Proof. See [68, Theorem 0.0.1].

Theorem 8.4 (Kedlaya). For $E \in \text{F-Isoc}^\dagger(X)$, the $K$-vector spaces $H^i_{\text{rig}}(X, E)$ are finite-dimensional for all $i \geq 0$ and zero for all $i > 2 \dim X$.

Proof. See [44, Theorem 1.2.1]. Alternatively, this can be deduced from Theorem 7.6 using the fact that Theorem 8.3 can be extended to logarithmic isocrystals (see [70]).
Remark 8.5. Theorem 8.4 fails for convergent $F$-isocrystals if $X$ is not proper: Theorem 8.3 (suitably stated) remains true without the properness condition, whereas crystalline cohomology for open varieties does not have good finiteness properties. More subtly, Theorem 8.4 also fails for overconvergent isocrystals without Frobenius structure, due to issues involving $p$-adic Liouville numbers (see Remark 5.9).

We have the following analogue of the Grothendieck-Ogg-Shafarevich formula for an overconvergent $F$-isocrystal on a curve.

**Theorem 8.6** (Christol–Mebkhout, Crew, Matsuda, Tsuzuki). Assume that $k$ is algebraically closed. Suppose that $X$ is geometrically irreducible of dimension 1, and let $\overline{X}$ be the smooth compactification of $X$. For $\mathcal{E} \in \mathbf{F-Isoc}^\dagger(X)$ and $x \in \overline{X} - X$, let $\operatorname{Swan}_x(\mathcal{E})$ denote the Swan conductor of the local monodromy representation of $\mathcal{E}$ at $x$ (Remark 4.11). Then
\[
\sum_{i=0}^{2} (-1)^i \dim_K H^i_{\text{rig}}(X, \mathcal{E}) = \chi(X) \text{ rank}(\mathcal{E}) - \sum_{x \in \overline{X} - X} \operatorname{Swan}_x(\mathcal{E}).
\]

**Proof.** See [45, Theorem 4.3.1].

Remark 8.7. There is also a theory of rigid cohomology with compact support admitting a form of Poincaré duality; see [44]. In terms of cohomology with compact support, the Lefschetz trace formula for Frobenius with coefficients in an overconvergent $F$-isocrystal holds for arbitrary (not necessarily smooth) varieties; see [28, Théorème 6.3], [45, (2.1.2)].

9. Theory of weights

Since rigid cohomology is a Weil cohomology theory, one may reasonably expect that the theory of weights in $\ell$-adic étale cohomology should carry over. This expectation turns out to be correct.

**Hypothesis 9.1.** Throughout §9, assume that $k = \mathbb{F}_q$ is finite, and fix an algebraic embedding $\iota : \overline{K} \hookrightarrow \mathbb{C}$.

**Definition 9.2.** For each finite extension $K'$ of $K$ within $\overline{K}$, let $\mathbf{F-Isoc}^\dagger(X) \otimes K'$ be the set of objects of $\mathbf{F-Isoc}^\dagger(X)$ equipped with a $K$-linear action of $K'$. Let $\mathbf{F-Isoc}^\dagger(X) \otimes \overline{K}$ be the direct 2-limit of the categories $\mathbf{F-Isoc}^\dagger(X) \otimes K'$ over all finite extensions $K'$ of $K$ within $\overline{K}$. Note that all of the previous results about $\mathbf{F-Isoc}^\dagger(X)$ can be formally promoted to $\mathbf{F-Isoc}^\dagger(X) \otimes K'$; we omit the details. (One twist is that the vertices of slope polygons need no longer have integral $y$-coordinates.)

For $\mathcal{F}$ a Weil $\overline{\mathbb{Q}}_p$-sheaf of rank 1 on $X$, the geometric monodromy group of $\mathcal{F}$ is always finite due to the mismatch between the $\ell$-adic and $p$-adic topologies [20, Proposition 1.3.4]. With a somewhat more intricate argument from geometric class field theory due to Katz–Lang [36], one obtains an analogous result in the $p$-adic case.

**Lemma 9.3** (Abe). For $n$ a positive integer, put $X_n = X \times_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and let $\pi_n : X_n \to X$ be the canonical projection. For any $\mathcal{E} \in \mathbf{F-Isoc}^\dagger(X) \otimes \overline{K}$, there exist a positive integer $n$ and an object $\mathcal{F} \in \mathbf{F-Isoc}^\dagger(\mathbb{F}_{q^n}) \otimes \overline{K}$ of rank 1 such that $\det(\pi_n^*\mathcal{E} \otimes \mathcal{F})$ is of finite order.

**Proof.** See [2, Lemma 6.1].
The following corollary is parallel to [21, 1.3].

**Corollary 9.4.** For any $\mathcal{E} \in \text{F-Isoc}^\dagger(X) \otimes \overline{K}$, there exist a positive integer $n$ and a decomposition
\begin{equation}
\mathcal{E} \cong \bigoplus_i \pi_{ns}(\mathcal{E}_i \otimes \mathcal{L}_i)
\end{equation}
in which for each $i$, $\mathcal{E}_i$ is an object of $\text{F-Isoc}^\dagger(X_n) \otimes \overline{K}$ with determinant of finite order and $\mathcal{L}_i$ is an object of $\text{F-Isoc}^\dagger(\mathbb{F}_q^n) \otimes \overline{K}$ of rank $1$. (Note that $n$ can be bounded above by $\text{rank}(\mathcal{E})$.)

**Definition 9.5.** For $n$ a positive integer, put $K_n = \text{Frac}W(\mathbb{F}_q^n) \subseteq \overline{K}$. An object of $\text{F-Isoc}^\dagger(\mathbb{F}_q^n) \otimes \overline{K}$ corresponds to a finite projective $(K_n \otimes_K \overline{K})$-module equipped with an isomorphism with its $(\varphi \otimes 1)$-pullback, or equivalently to a finite-dimensional $\overline{K}$-vector space equipped with an invertible endomorphism (the linearized Frobenius action).

Suppose now that $\mathcal{E} \in \text{F-Isoc}^\dagger(X)$.

- For $w \in \mathbb{R}$, we say that $\mathcal{E}$ is $\nu$-pure of weight $w$ if for each closed point $x \in X$ with residue field $\mathbb{F}_q^n$, each eigenvalue $\alpha$ of the linearized Frobenius action on $\mathcal{E}_x$ satisfies $|\alpha| = q^{nw/2}$.
- We say that $\mathcal{E}$ is $\nu$-mixed of weights $\geq w$ (resp. $\leq w$) if it is a successive extension of objects, each of which is $\nu$-pure of some weight $\geq w$ (resp. $\leq w$).
- We say that $\mathcal{E}$ is $\nu$-real if for each $x$, the coefficients of the characteristic polynomial of the linearized Frobenius on $\mathcal{E}_x$ map into $\mathbb{R}$ via $\nu$; we say that $\mathcal{E}$ is $\nu$-realizable if it occurs as a direct summand of an $\nu$-real object. (Both of these conditions only depend on $\mathcal{E}$ up to semisimplification.)

We have the following partial analogue of Deligne’s “Weil II” theorem [20]. A more complete analogue can be stated in terms of constructible coefficients; see §10.

**Theorem 9.6** (Kedlaya). Suppose that $\mathcal{E} \in \text{F-Isoc}^\dagger(X) \otimes \overline{K}$ is $\nu$-mixed of weights $\geq w$. Then for all $i \geq 0$, $H^i_{\text{rig}}(X, \mathcal{E})$ is $\nu$-mixed of weights $\geq w + i$.

**Proof.** We may reduce to the case $\mathcal{E} \in \text{F-Isoc}^\dagger(X)$, for which see [45, Theorem 5.3.2].

**Corollary 9.7.** Let
\begin{equation}
0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0
\end{equation}
be an exact sequence in $\text{F-Isoc}^\dagger(X) \otimes \overline{K}$ in which $\mathcal{E}_i$ is $\nu$-pure of weight $w_i$, $w_1 \neq w_2$, and $w_2 < w_1 + 1$. (In particular, these conditions hold if $w_2 < w_1$.) Then this sequence splits in $\text{F-Isoc}^\dagger(X) \otimes \overline{K}$.

**Proof.** We reduce formally to the case where the exact sequence is in $\text{F-Isoc}^\dagger(X)$. We have the following exact sequence of Hochschild-Serre type:
\begin{equation}
0 \to H^0_{\text{rig}}(X, \mathcal{E}_2^\vee \otimes \mathcal{E}_1)_F \to \text{Ext}^1_{\text{F-Isoc}^\dagger(X)}(\mathcal{E}_2, \mathcal{E}_1) \to H^1_{\text{rig}}(X, \mathcal{E}_2^\vee \otimes \mathcal{E}_1)_F.
\end{equation}
In this sequence, $H^0_{\text{rig}}(X, \mathcal{E}_2^\vee \otimes \mathcal{E}_1)$ is finite-dimensional and $\nu$-pure of weight $w_1 - w_2 \neq 0$, so its Frobenius coinvariants are trivial. Meanwhile, by Theorem 9.6, $H^1_{\text{rig}}(X, \mathcal{E}_2^\vee \otimes \mathcal{E}_1)$ is $\nu$-mixed of weights $\geq w_1 - w_2 + 1 > 0$, so its Frobenius invariants are also trivial. □
Corollary 9.8 (Abe-Caro). Any \( E \in F\text{-Isoc}^\dagger(X) \) which is \( \iota \)-mixed admits a unique filtration
\[
0 = E_0 \subset \cdots \subset E_i = E
\]
such that each successive quotient \( E_i/E_{i-1} \) is \( \iota \)-pure of some weight \( w_i \), and \( w_1 < \cdots < w_l \). We call this the weight filtration of \( E \).

Proof. This is immediate from Corollary 9.7. For an independent derivation (and an extension to complexes), see [4, Theorem 4.3.4]. □

Remark 9.9. In Corollary 9.7, half of the proof applies in the case \( w_1 = w_2 \): the extension class in \( H^1_{\text{rig}}(X, E_2^* \otimes E_1)^F \) still vanishes. We thus still get a splitting in the category of overconvergent isocrystals without Frobenius structures; consequently, any \( \iota \)-pure object in \( F\text{-Isoc}^\dagger(X) \otimes \overline{K} \) becomes semisimple in the category of overconvergent isocrystals without Frobenius structure.

Remark 9.10. While the proof of Theorem 9.6 draws many elements from Deligne’s original arguments in [20], in overall form it more closely resembles the stationary phase method of Laumon [61], and even more closely the exposition of Katz [35] which makes some minor simplifications to Laumon’s treatment. In fact, translating the arguments from [45] back to the \( \ell \)-adic side would yield an argument differing slightly even from [35].

One pleasing feature of the \( p \)-adic approach is that the \( \ell \)-adic Fourier transform analogizes to a Fourier transform on some sort of \( \mathcal{D} \)-modules on the affine line, which is genuinely constructed by interchanging terms in a Weyl algebra. This point of view was originally developed by Huyghe [32], and is maintained in [45].

The following is analogous to a statement in the \( \ell \)-adic case which is a consequence of the Chebotarev density theorem; however, here one must instead make an argument using weights.

Theorem 9.11 (Tsuzuki). Suppose that \( E_1, E_2 \in F\text{-Isoc}^\dagger(X) \otimes \overline{K} \) are \( \iota \)-mixed and have the same set of Frobenius eigenvalues at each closed point \( x \in X \). Then \( E_1, E_2 \) have the same semisimplification in \( F\text{-Isoc}^\dagger(X) \otimes \overline{K} \).

Proof. See [3, Proposition A.3.1]. □

Remark 9.12. By analogy with Deligne’s equidistribution theorem, one has an equidistribution theorem for Frobenius conjugacy classes in rigid cohomology; this was described explicitly by Crew in the case where \( \dim(X) = 1 \) [15, Theorem 10.11], but in light of the general theory of weights, one can adapt the proof of [20, Théorème 3.5.3] to arbitrary \( X \). We omit further details here.

10. A REMARK ON CONSTRUCTIBLE COEFFICIENTS

To get any further in the study of rigid cohomology, one needs an analogue not just of lisse étale sheaves, but also constructible étale sheaves. Berthelot originally proposed a theory of arithmetic \( \mathcal{D} \)-modules for this purpose [9], and conjectured that holonomic objects in this theory (equipped with Frobenius structure) are stable under the six operations formalism. This result remains unknown, partly because the definition of holonomicity is itself a bit subtle; for instance, a direct arithmetic analogue of Bernstein’s inequality fails, so one must use Frobenius descent to salvage it.
In the interim, a modified definition of overholonomic arithmetic $\mathcal{D}$-modules has been given by Caro [10], as a way to formally salvage the six operations formalism. Of course, this provides little benefit unless one can prove that this category contains the overconvergent $F$-isocrystals as a full subcategory; fortunately, this is known thanks to a difficult theorem of Caro and Tsuzuki [11] (whose proof makes essential use of Theorem 7.6). The theory of weights in Caro’s formalism is developed in [4].

Recently, Le Stum has given a site-theoretic construction of overconvergent $F$-isocrystals [62] and proposed a theory of constructible isocrystals [63]. It is hoped that this again yields a six operations formalism, with somewhat less technical baggage required than in Caro’s approach.

In any case, using these methods, Abe [3] has recently succeeded in porting L. Lafforgue’s proof of the Langlands correspondence for $GL_n$ over a function field [59] into $p$-adic cohomology; this immediately resolves Deligne’s conjecture on crystalline companions [20, Conjecture 1.2.10] in dimension 1, and also gives some results in higher dimension. See [58] for further discussion.

11. FURTHER READING

We conclude with some suggestions for additional reading, in addition to the references already cited.

- Berthelot’s first sketch of the theory of rigid cohomology is the article [7]; while quite dated, it remains a wonderfully readable introduction to the circle of ideas underpinning the subject.
- In [48], there is a discussion of $p$-adic cohomology oriented towards machine computations, especially of zeta functions.
- In [51], some discussion is given of how recent (circa 2009) results in rigid cohomology tie back to older results in crystalline cohomology.

APPENDIX A. A FEW PROOFS

In this appendix, we record some proofs which are variants of those given in [24]. We start with one of the steps in the proof of Theorem 5.5.

**Lemma A.1.** The functor $F\text{-Isoc}(X, Y) \to F\text{-Isoc}(X)$ is fully faithful.

**Proof.** We proceed by emulating the argument used in [40, Theorem 1.1] to reduce to the local model statement described in Remark 5.8, starting with some geometric simplifications using Remark 2.8.

Choose a compactification $\overline{Y}$ of $Y$; we may assume without loss of generality that $X$ is dense in $Y$, $Y$ is dense in $\overline{Y}$, and $\dim X = \dim \overline{Y}$; let $n$ be the common dimension.

By Remark 2.8, we may cover $X$ with affine open subspaces $U$, each of which has the property that there is a finite morphism $\overline{Y} \to \mathbb{P}^n_k$ étale over $\mathbb{A}^n_k$ such that $U = \overline{Y} \times_{\mathbb{P}^n_k} \mathbb{A}^n_k$. Since we may check the claim locally on $X$ (without changing $Y$), we may assume that $X = U$. Using Theorem 5.1, we may shrink $Y$ to ensure that it is finite flat over an open subscheme of $\mathbb{P}^n_k$. We may then invoke pushforward functoriality (Remark 2.7) to reduce to the case where $X = \mathbb{A}^n_k$ and $Y \subseteq \mathbb{P}^n_k$.

Let $H$ be the hyperplane at infinity in $\mathbb{P}^n_k$. If at this point $Y$ does not contain a dense subset of $H$, then we may invoke Theorem 5.1 to reduce to the case $X = Y$, which is trivial.
We may thus assume the contrary, in which case we may invoke Theorem 5.1 to reduce to the case $X = \mathbb{A}_k^n, Y = \mathbb{P}_k^n$.

Once more working locally on $Y$, we may further reduce to the case where $Y$ is an affine space and $X$ is the complement of a coordinate hyperplane. Fixing coordinates, we take

$$X = \text{Spec } k[x_1, \ldots, x_n, x_1^{-1}], \quad Y = \text{Spec } k[x_1, \ldots, x_n].$$

Using internal Homs, we further reduce to checking that for $X \in \mathbf{F} \text{-Isoc}(X, Y)$, any morphism in $\mathbf{F} \text{-Isoc}(X)$ from the constant object to $X$ descends uniquely to $\mathbf{F} \text{-Isoc}(X, Y)$.

Let $S_2$ be the ring obtained from the $p$-adic completion of $W(k)[x_1, \ldots, x_n, x_1^{-1}]$ by inverting $p$. Let $\sigma : S_2 \to S_2$ be the Frobenius lift taking $x_i$ to $x_i^p$. Let $S_1$ be the subring of $S_2$ obtained by taking the unions of the $p$-adic completions of $W(k)[x_1, \ldots, x_n, px_1^{-m}]$ over all $m > 0$, then inverting $p$. We may then represent $X$ as a finite projective module $M$ over $S_1$ equipped with compatible semilinear actions of $\sigma$ and $\nabla$, and a morphism in $\mathbf{F} \text{-Isoc}(X)$ from the constant object to $X$ is an element $v$ of $M \otimes_{S_1} S_2$ for which $\sigma(v) = v$ and $\nabla(v) = 0$. To complete the proof, we must establish that in fact $v \in M$. By passing from $k$ to $k((x_2, \ldots, x_n)_{\text{perf}}$, we may further reduce to the case $n = 1$; we may then apply the local model statement [40, Theorem 5.1] to deduce that $v \in M \otimes_{S_1} S_3$ for $S_3 = W(k)[x_1][p^{-1}]$.

Let $S_4$ be the ring obtained from the $p$-adic completion of $W(k)((x_1))$ by inverting $p$; within $S_4$ we have $S_2 \cap S_3 = S_1$. Since $M$ is a direct summand of a free $S_1$-module, we may deduce that

$$v \in (M \otimes_{S_1} S_2) \cap (M \otimes_{S_1} S_3) = M \otimes_{S_1} (S_2 \cap S_3) = M \otimes_{S_1} S_1 = M.$$  

This completes the proof that $\mathbf{F} \text{-Isoc}(X, Y) \to \mathbf{F} \text{-Isoc}(X)$ is fully faithful, as required. □

We next give an alternate approach to Theorem 6.3 in the case $X \in \mathbf{F} \text{-Isoc}(X)$ based on reduction to the local model statement, which is an unpublished result from the author’s PhD thesis [37, Theorem 5.2.1].

**Lemma A.2.** Let

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

be a short exact sequence in $\mathbf{F} \text{-Isoc}(k((t)))$ with $\mathcal{E}_i$ isoclinic of slope $s_i$ and $s_2 - s_1 > 1$. Then this sequence splits uniquely.

**Proof.** Using internal Homs, we may reduce to treating the case where $\mathcal{E}_2$ is trivial, and in particular $s_2 = 0$ and $s_1 < -1$. The extension group $\text{Ext}^1_{\mathbf{F} \text{-Isoc}(k((t)))}(\mathcal{E}_2, \mathcal{E}_1)$ may then be computed as the first cohomology group of the double complex

$$\begin{array}{ccc}
\mathcal{E}_1 & \xrightarrow{d/dt} & \mathcal{E}_1 \\
\downarrow{\sigma - 1} & & \downarrow{pt^{-1} \sigma - 1} \\
\mathcal{E}_1 & \xrightarrow{d/dt} & \mathcal{E}_1
\end{array}$$

where the top left entry is placed in degree 0. A 1-cocycle is therefore a pair $(v_1, v_2) \in \mathcal{E}_1 \times \mathcal{E}_1$ with $d_{\mathcal{E}_1}(v_1) = (pt^{-1} \sigma - 1)(v_2)$, and a 1-coboundary is a pair for which there exists an element $v \in \mathcal{E}_1$ with $(\sigma - 1)(v) = v_1, d_{\mathcal{E}_1}(v) = v_2$. 

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For $c > 0$, let $\Gamma_{\text{perf}(c)}$ be the subring of $\Gamma_{\text{perf}}$ consisting of those $x$ for which for each $n \geq 0$, there exists $y_n \in \Gamma$ such that $\sigma^{-n}(y_n) - x$ is divisible by $p^{[c]n}$. Note that for $c > 1$, the operator $\frac{d}{dt}$ on $\Gamma$ extends to a well-defined map $\Gamma_{\text{perf}(c)}[p^{-1}] \to \Gamma_{\text{perf}(c-1)}[p^{-1}]$.

Since $\mathcal{E}_1$ is isoclinic of slope $s_1 < -1$, we may define
\[
v = \sigma(1 + \sigma^{-1} + \sigma^{-2} + \cdots)(v_1) \in \mathcal{E}_1 \otimes_{\Gamma[p^{-1}]} \Gamma_{\text{perf}(-s_1)}[p^{-1}]\]
via a convergent infinite series. By the previous paragraph, we may then form $\frac{d}{dt}(v) \in \mathcal{E}_1 \otimes_{\Gamma[p^{-1}]} \Gamma_{\text{perf}(-s_1-1)}[p^{-1}]$, which satisfies
\[
(pt^{p-1} - 1) \left( \frac{d}{dt}(v) - v_2 \right) = 0.
\]
Since $s_1 + 1 < 0$, $pt^{p-1} - 1$ is bijective on $\mathcal{E}_1 \otimes_{\Gamma[p^{-1}]} \Gamma_{\text{perf}}[p^{-1}]$, so this forces
\[
\frac{d}{dt}(v) = v_2.
\]
It will now suffice to check that this equality forces $v \in \mathcal{E}_1$.

To see this, write $\Gamma_{\text{perf}}[p^{-1}]$ as a completed direct sum of $t^\alpha \Gamma[p^{-1}]$ with $\alpha$ varying over $\mathbb{Z}[p^{-1}] \cap [0, 1)$, then split $\mathcal{E}_1 \otimes_{\Gamma[p^{-1}]} \Gamma_{\text{perf}}[p^{-1}]$ accordingly. For each component $t^\alpha v_\alpha$ of $v$ with $\alpha \neq 0$, (A.2.1) then implies $\frac{d}{dt}(t^\alpha v_\alpha) = 0$.

Now let $\Gamma_{\text{unr}}$ be the completion of the maximal unramified extension of $\Gamma$; the derivation $\frac{d}{dt}$ extends uniquely by continuity to $\Gamma_{\text{unr}}$. By a suitably precise form of Theorem 3.7 (e.g., see [74, Corollary 5.1.4]), there exists a basis $e_1, \ldots, e_m$ of $\mathcal{E}_1 \otimes_{\Gamma[p^{-1}]} \Gamma_{\text{unr}}[p^{-1}]$ such that $\frac{d}{dt}(e_i) = 0$ for $i = 1, \ldots, n$. Writing $v_\alpha = \sum_{i=1}^m c_i e_i$ with $c_i \in \Gamma_{\text{unr}}[p^{-1}]$, we have
\[
0 = \frac{d}{dt}(t^\alpha v_\alpha) = \sum_{i=1}^m t^\alpha \left( \alpha t^{-1} c_i + \frac{dc_i}{dt} \right) e_i.
\]
However, the $p$-adic valuation of $\alpha$ is negative and the $p$-adic valuation of $c_i$ is no greater than that of its derivative, so (A.2.2) can only hold if $c_i = 0$ for all $i = 0$. This implies that $v \in \mathcal{E}_1$, as needed. \hfill \Box

**Lemma A.3.** Theorem 6.3 holds in the case $\mathcal{E} \in \text{F-Isoc}(X)$.

**Proof of Theorem 6.3.** We first show that the claim may be reduced from $X$ to an open dense affine subspace $U$. The splitting of $\mathcal{E}$ is defined by a projector, so it can be extended from $U$ to $X$ using Theorem 5.5. This in turn implies (a) using Theorem 3.12: the sum of the slopes of $\mathcal{E}_1$ is locally constant, the largest slope of $\mathcal{E}_1$ can only decrease under specialization, and the smallest slope of $\mathcal{E}_2$ can only increase under specialization.

Using Theorem 3.12 again, we may thus reduce to the case where $\mathcal{E}$ has constant slope polygon (so we no longer need to verify (a) separately). By Corollary 4.2, $\mathcal{E}$ now admits a slope filtration. We are thus reduced to showing that if $X$ is affine and
\[
0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0
\]
is a short exact sequence in $\text{F-Isoc}(X)$ with $\mathcal{E}_1$ isoclinic of slope $s_1$ and $s_2 - s_1 > 1$, then this sequence splits uniquely. Using Remark 2.7 and Remark 2.8, we reduce to the case $X = A^n_k$ (this is not essential but makes the argument slightly more transparent). As in Definition 2.1, we may realize $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ as finite projective modules over the Tate algebra $R = K(T_1, \ldots, T_n)$ equipped with compatible actions of the standard Frobenius $\sigma : T_i \mapsto T_i^p$ and $\nabla$. Let $R'$ be
the completion of $K(T_1, \ldots, T_n)[T_1^{1/p\infty}, \ldots, T_n^{1/p\infty}]$ for the Gauss norm; then the sequence of $\sigma$-modules splits uniquely over $R'$, and we must show that this splitting descends to $R$ and is compatible with the action of the derivations $\frac{d}{dT_1}, \ldots, \frac{d}{dT_n}$. For this, we may apply Lemma A.2 to treat each variable individually.

\[\square\]

REFERENCES


