A–1 How many primes among the positive integers, written as usual in base 10, are alternating 1’s and 0’s, beginning and ending with 1?

A–2 Evaluate \( \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx \) where \( a \) and \( b \) are positive.

A–3 Prove that if \( \alpha \) is an irrational number, \( 0 < \alpha < 1 \), there is a finite game with an honest coin such that the probability of one player winning the game is \( \alpha \)? (An honest coin is one for which the probability of heads and the probability of tails are both \( \frac{1}{2} \). A game is finite if with probability 1 it must end in a finite number of moves.)

A–4 If \( \alpha \) is an irrational number, \( 0 < \alpha < 1 \), is there a finite game with an honest coin such that the probability of one player winning the game is \( \alpha \)? (An honest coin is one for which the probability of heads and the probability of tails are both \( \frac{1}{2} \). A game is finite if with probability 1 it must end in a finite number of moves.)

A–5 Let \( m \) be a positive integer and let \( \mathcal{G} \) be a regular \((2m+1)\)-gon inscribed in the unit circle. Show that there is a positive constant \( A \), independent of \( m \), with the following property. For any points \( p \) inside \( \mathcal{G} \) there are two distinct vertices \( v_1 \) and \( v_2 \) of \( \mathcal{G} \) such that

\[
| |p - v_1| - |p - v_2| | < \frac{1}{m} - \frac{A}{m^2}.
\]

Here \( |s - t| \) denotes the distance between the points \( s \) and \( t \).

A–6 Let \( \alpha = 1 + a_1x + a_2x^2 + \cdots \) be a formal power series with coefficients in the field of two elements. Let

\[
a_n = \begin{cases} 
 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block} \\
 0 & \text{otherwise}.
\end{cases}
\]

(For example, \( a_{36} = 1 \) because \( 36 = 100100_2 \) and \( a_{20} = 0 \) because \( 20 = 10100_2 \).) Prove that \( \alpha^3 + x^4 \alpha + 1 = 0 \).

B–2 Let \( S \) be a non-empty set with an associative operation that is left and right cancellative (\( xy = xz \) implies \( y = z \) and \( yx = zx \) implies \( y = z \)). Assume that for every \( a \) in \( S \) the set \( \{a^n : n = 1, 2, 3, \ldots \} \) is finite. Must \( S \) be a group?

B–3 Let \( f \) be a function on \([0, \infty)\), differentiable and satisfying

\[
f'(x) = -3f(x) + 6f(2x)
\]

for \( x > 0 \). Assume that \( |f(x)| \leq e^{-\sqrt{x}} \) for \( x \geq 0 \) (so that \( f(x) \) tends rapidly to 0 as \( x \) increases). For \( n \) a non-negative integer, define

\[
\mu_n = \int_0^\infty x^n f(x) \, dx
\]

(sometimes called the \( n \)th moment of \( f \)).

a) Express \( \mu_n \) in terms of \( \mu_0 \).

b) Prove that the sequence \( \{\mu_n^{1/n}\} \) always converges, and that the limit is 0 only if \( \mu_0 = 0 \).

B–4 Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

B–5 Label the vertices of a triangle \( T \) (quadrilateral with two parallel sides) inscribed in the unit circle as \( A, B, C, D \) so that \( AB \) is parallel to \( CD \) and \( A, B, C, D \) are in counterclockwise order. Let \( s_1, s_2, \) and \( d \) denote the lengths of the line segments \( AB, CD, \) and \( OE \), where \( E \) is the point of intersection of the diagonals of \( T \), and \( O \) is the center of the circle. Determine the least upper bound of \( \frac{\sqrt{s_1^2 + s_2^2}}{d} \) over all such \( T \) for which \( d \neq 0 \), and describe all cases, if any, in which it is attained.

B–6 Let \( (x_1, x_2, \ldots, x_n) \) be a point chosen at random from the \( n \)-dimensional region defined by \( 0 < x_1 < x_2 < \cdots < x_n < 1 \). Let \( f \) be a continuous function on \([0, 1]\) with \( f(1) = 0 \). Set \( x_0 = 0 \) and \( x_{n+1} = 1 \). Show that the expected value of the Riemann sum

\[
\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})
\]

is \( \int_0^1 f(t)P(t) \, dt \), where \( P \) is a polynomial of degree \( n \), independent of \( f \), with \( 0 \leq P(t) \leq 1 \) for \( 0 \leq t \leq 1 \).