The 55th William Lowell Putnam Mathematical Competition
Saturday, December 3, 1994

A–1 Suppose that a sequence $a_1, a_2, a_3, \dots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^\infty a_n$ diverges.

A–2 Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

A–3 Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color whch are at least a distance $2 - \sqrt{2}$ apart.

A–4 Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

A–5 Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}},$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a,b) contains a nonempty subinterval (c,d) that does not intersect S.

A–6 Let $f_1, \dots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathscr{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}\},$$

$e_i = 0$ or 1 for $1 \leq i \leq 10$. (f_i^0 is the identity function and $f_i^1 = f_i$.) Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathscr{F}$ map A to itself.

B–1 Find all positive integers n that are within 250 of exactly 15 perfect squares.

B–2 For which real numbers c is there a straight line that intersects the curve

$$x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

B–3 Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

B–4 For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n \to \infty} d_n = \infty$.

B–5 For any real number α, define the function $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

B–6 For any integer n, set

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a,b\} = \{c,d\}$.