

**Solutions to the 60th William Lowell Putnam Mathematical Competition**  
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A-1 Note that if  $r(x)$  and  $s(x)$  are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if  $F(x)$  is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x + 3|)/2 \\ &\quad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x + 3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set  $f(x) = (3x + 3)/2$ ,  $g(x) = 5x/2$ , and  $h(x) = -x + \frac{1}{2}$ .

A-2 First solution: First factor  $p(x) = q(x)r(x)$ , where  $q$  has all real roots and  $r$  has all complex roots. Notice that each root of  $q$  has even multiplicity, otherwise  $p$  would have a sign change at that root. Thus  $q(x)$  has a square root  $s(x)$ .

Now write  $r(x) = \prod_{j=1}^k (x - a_j)(x - \bar{a}_j)$  (possible because  $r$  has roots in complex conjugate pairs). Write  $\prod_{j=1}^k (x - a_j) = t(x) + iu(x)$  with  $t, x$  having real coefficients. Then for  $x$  real,

$$\begin{aligned} p(x) &= q(x)r(x) \\ &= s(x)^2(t(x) + iu(x))\overline{(t(x) + iu(x))} \\ &= (s(x)t(x))^2 + (s(x)u(x))^2. \end{aligned}$$

(Alternatively, one can factor  $r(x)$  as a product of quadratic polynomials with real coefficients, write each as a sum of squares, then multiply together to get a sum of many squares.)

Second solution: We proceed by induction on the degree of  $p$ , with base case where  $p$  has degree 0. As in the first solution, we may reduce to a smaller degree in case  $p$  has any real roots, so assume it has none. Then  $p(x) > 0$  for all real  $x$ , and since  $p(x) \rightarrow \infty$  for  $x \rightarrow \pm\infty$ ,  $p$  has a minimum value  $c$ . Now  $p(x) - c$  has real roots, so as above, we deduce that  $p(x) - c$  is a sum of squares. Now add one more square, namely  $(\sqrt{c})^2$ , to get  $p(x)$  as a sum of squares.

A-3 First solution: Computing the coefficient of  $x^{n+1}$  in the identity  $(1 - 2x - x^2) \sum_{m=0}^{\infty} a_m x^m = 1$  yields the recurrence  $a_{n+1} = 2a_n + a_{n-1}$ ; the sequence  $\{a_n\}$  is then characterized by this recurrence and the initial conditions  $a_0 = 1, a_1 = 2$ .

Define the sequence  $\{b_n\}$  by  $b_{2n} = a_{n-1}^2 + a_n^2$ ,  $b_{2n+1} =$

$a_n(a_{n-1} + a_{n+1})$ . Then

$$\begin{aligned} 2b_{2n+1} + b_{2n} &= 2a_n a_{n+1} + 2a_{n-1} a_n + a_{n-1}^2 + a_n^2 \\ &= 2a_n a_{n+1} + a_{n-1} a_{n+1} + a_n^2 \\ &= a_{n+1}^2 + a_n^2 = b_{2n+2}, \end{aligned}$$

and similarly  $2b_{2n} + b_{2n-1} = b_{2n+1}$ , so that  $\{b_n\}$  satisfies the same recurrence as  $\{a_n\}$ . Since further  $b_0 = 1, b_1 = 2$  (where we use the recurrence for  $\{a_n\}$  to calculate  $a_{-1} = 0$ ), we deduce that  $b_n = a_n$  for all  $n$ . In particular,  $a_n^2 + a_{n+1}^2 = b_{2n+2} = a_{2n+2}$ .

Second solution: Note that

$$\begin{aligned} \frac{1}{1 - 2x - x^2} &= \frac{1}{2\sqrt{2}} \left( \frac{\sqrt{2} + 1}{1 - (1 + \sqrt{2})x} + \frac{\sqrt{2} - 1}{1 - (1 - \sqrt{2})x} \right) \end{aligned}$$

and that

$$\frac{1}{1 + (1 \pm \sqrt{2})x} = \sum_{n=0}^{\infty} (1 \pm \sqrt{2})^n x^n,$$

so that

$$a_n = \frac{1}{2\sqrt{2}} \left( (\sqrt{2} + 1)^{n+1} - (1 - \sqrt{2})^{n+1} \right).$$

A simple computation (omitted here) now shows that  $a_n^2 + a_{n+1}^2 = a_{2n+2}$ .

Third solution (by Richard Stanley): Let  $A$  be the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ . A simple induction argument shows that

$$A^{n+2} = \begin{pmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{pmatrix}.$$

The desired result now follows from comparing the top left corner entries of the equality  $A^{n+2} A^{n+2} = A^{2n+4}$ .

A-4 Denote the series by  $S$ , and let  $a_n = 3^n/n$ . Note that

$$\begin{aligned} S &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m(a_m + a_n)} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_n(a_m + a_n)}, \end{aligned}$$

where the second equality follows by interchanging  $m$

and  $n$ . Thus

$$\begin{aligned} 2S &= \sum_m \sum_n \left( \frac{1}{a_m(a_m + a_n)} + \frac{1}{a_n(a_m + a_n)} \right) \\ &= \sum_m \sum_n \frac{1}{a_m a_n} \\ &= \left( \sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2. \end{aligned}$$

But

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$$

since, e.g., it's  $f'(1)$ , where

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{3}{3-x},$$

and we conclude that  $S = 9/32$ .

A-5 First solution: (by Reid Barton) Let  $r_1, \dots, r_{1999}$  be the roots of  $P$ . Draw a disc of radius  $\varepsilon$  around each  $r_i$ , where  $\varepsilon < 1/3998$ ; this disc covers a subinterval of  $[-1/2, 1/2]$  of length at most  $2\varepsilon$ , and so of the 2000 (or fewer) uncovered intervals in  $[-1/2, 1/2]$ , one, which we call  $I$ , has length at least  $\delta = (1 - 3998\varepsilon)/2000 > 0$ . We will exhibit an explicit lower bound for the integral of  $|P(x)|/P(0)$  over this interval, which will yield such a bound for the entire integral.

Note that

$$\frac{|P(x)|}{|P(0)|} = \prod_{i=1}^{1999} \frac{|x - r_i|}{|r_i|}.$$

Also note that by construction,  $|x - r_i| \geq \varepsilon$  for each  $x \in I$ . If  $|r_i| \leq 1$ , then we have  $\frac{|x - r_i|}{|r_i|} \geq \varepsilon$ . If  $|r_i| > 1$ , then

$$\frac{|x - r_i|}{|r_i|} = |1 - x/r_i| \geq 1 - |x/r_i| \geq 1/2 > \varepsilon.$$

We conclude that  $\int_I |P(x)/P(0)| dx \geq \delta\varepsilon$ , independent of  $P$ .

Second solution: It will be a bit more convenient to assume  $P(0) = 1$  (which we may achieve by rescaling unless  $P(0) = 0$ , in which case there is nothing to prove) and to prove that there exists  $D > 0$  such that  $\int_{-1}^1 |P(x)| dx \geq D$ , or even such that  $\int_0^1 |P(x)| dx \geq D$ .

We first reduce to the case where  $P$  has all of its roots in  $[0, 1]$ . If this is not the case, we can factor  $P(x)$  as  $Q(x)R(x)$ , where  $Q$  has all roots in the interval and  $R$  has none. Then  $R$  is either always positive or always negative on  $[0, 1]$ ; assume the former. Let  $k$  be the largest positive real number such that  $R(x) - kx \geq 0$  on  $[0, 1]$ ; then

$$\begin{aligned} \int_{-1}^1 |P(x)| dx &= \int_{-1}^1 |Q(x)R(x)| dx \\ &> \int_{-1}^1 |Q(x)(R(x) - kx)| dx, \end{aligned}$$

and  $Q(x)(R(x) - kx)$  has more roots in  $[0, 1]$  than does  $P$  (and has the same value at 0). Repeating this argument shows that  $\int_0^1 |P(x)| dx$  is greater than the corresponding integral for some polynomial with all of its roots in  $[0, 1]$ .

Under this assumption, we have

$$P(x) = c \prod_{i=1}^{1999} (x - r_i)$$

for some  $r_i \in (0, 1]$ . Since

$$P(0) = -c \prod r_i = 1,$$

we have

$$|c| \geq \prod |r_i^{-1}| \geq 1.$$

Thus it suffices to prove that if  $Q(x)$  is a *monic* polynomial of degree 1999 with all of its roots in  $[0, 1]$ , then  $\int_0^1 |Q(x)| dx \geq D$  for some constant  $D > 0$ . But the integral of  $\int_0^1 \prod_{i=1}^{1999} |x - r_i| dx$  is a continuous function for  $r_i \in [0, 1]$ . The product of all of these intervals is compact, so the integral achieves a minimum value for some  $r_i$ . This minimum is the desired  $D$ .

Third solution (by Abe Kunin): It suffices to prove the stronger inequality

$$\sup_{x \in [-1, 1]} |P(x)| \leq C \int_{-1}^1 |P(x)| dx$$

holds for some  $C$ . But this follows immediately from the following standard fact: any two norms on a finite-dimensional vector space (here the polynomials of degree at most 1999) are equivalent. (The proof of this statement is also a compactness argument:  $C$  can be taken to be the maximum of the L1-norm divided by the sup norm over the set of polynomials with L1-norm 1.)

Note: combining the first two approaches gives a constructive solution with a constant that is better than that given by the first solution, but is still far from optimal. I don't know offhand whether it is even known what the optimal constant and/or the polynomials achieving that constant are.

A-6 Rearranging the given equation yields the much more tractable equation

$$\frac{a_n}{a_{n-1}} = 6 \frac{a_{n-1}}{a_{n-2}} - 8 \frac{a_{n-2}}{a_{n-3}}.$$

Let  $b_n = a_n/a_{n-1}$ ; with the initial conditions  $b_2 = 2, b_3 = 12$ , one easily obtains  $b_n = 2^{n-1}(2^{n-2} - 1)$ , and so

$$a_n = 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1).$$

To see that  $n$  divides  $a_n$ , factor  $n$  as  $2^k m$ , with  $m$  odd. Then note that  $k \leq n \leq n(n-1)/2$ , and that there exists  $i \leq m-1$  such that  $m$  divides  $2^i - 1$ , namely  $i = \phi(m)$  (Euler's totient function: the number of integers in  $\{1, \dots, m\}$  relatively prime to  $m$ ).

B-1 The answer is  $1/3$ . Let  $G$  be the point obtained by reflecting  $C$  about the line  $AB$ . Since  $\angle ADC = \frac{\pi-\theta}{2}$ , we find that  $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$ , so that  $E, D, G$  are collinear. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

where we have used the law of sines in  $\triangle BDG$ . But by l'Hôpital's Rule,

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta \rightarrow 0} \frac{\cos(\theta/2)}{3 \cos(3\theta/2)} = 1/3.$$

B-2 First solution: Suppose that  $P$  does not have  $n$  distinct roots; then it has a root of multiplicity at least 2, which we may assume is  $x = 0$  without loss of generality. Let  $x^k$  be the greatest power of  $x$  dividing  $P(x)$ , so that  $P(x) = x^k R(x)$  with  $R(0) \neq 0$ ; a simple computation yields

$$P''(x) = (k^2 - k)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x).$$

Since  $R(0) \neq 0$  and  $k \geq 2$ , we conclude that the greatest power of  $x$  dividing  $P''(x)$  is  $x^{k-2}$ . But  $P(x) = Q(x)P''(x)$ , and so  $x^2$  divides  $Q(x)$ . We deduce (since  $Q$  is quadratic) that  $Q(x)$  is a constant  $C$  times  $x^2$ ; in fact,  $C = 1/(n(n-1))$  by inspection of the leading-degree terms of  $P(x)$  and  $P''(x)$ .

Now if  $P(x) = \sum_{j=0}^n a_j x^j$ , then the relation  $P(x) = Cx^2 P''(x)$  implies that  $a_j = Cj(j-1)a_j$  for all  $j$ ; hence  $a_j = 0$  for  $j \leq n-1$ , and we conclude that  $P(x) = a_n x^n$ , which has all identical roots.

Second solution (by Greg Kuperberg): Let  $f(x) = P''(x)/P(x) = 1/Q(x)$ . By hypothesis,  $f$  has at most two poles (counting multiplicity).

Recall that for any complex polynomial  $P$ , the roots of  $P'$  lie within the convex hull of  $P$ . To show this, it suffices to show that if the roots of  $P$  lie on one side of a line, say on the positive side of the imaginary axis, then  $P'$  has no roots on the other side. That follows because if  $r_1, \dots, r_n$  are the roots of  $P$ ,

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - r_i}$$

and if  $z$  has negative real part, so does  $1/(z - r_i)$  for  $i = 1, \dots, n$ , so the sum is nonzero.

The above argument also carries through if  $z$  lies on the imaginary axis, provided that  $z$  is not equal to a root of

$P$ . Thus we also have that no roots of  $P'$  lie on the sides of the convex hull of  $P$ , unless they are also roots of  $P$ .

From this we conclude that if  $r$  is a root of  $P$  which is a vertex of the convex hull of the roots, and which is not also a root of  $P'$ , then  $f$  has a single pole at  $r$  (as  $r$  cannot be a root of  $P''$ ). On the other hand, if  $r$  is a root of  $P$  which is also a root of  $P'$ , it is a multiple root, and then  $f$  has a double pole at  $r$ .

If  $P$  has roots not all equal, the convex hull of its roots has at least two vertices.

B-3 We first note that

$$\sum_{m,n>0} x^m y^n = \frac{xy}{(1-x)(1-y)}.$$

Subtracting  $S$  from this gives two sums, one of which is

$$\sum_{m \geq 2n+1} x^m y^n = \sum_n y^n \frac{x^{2n+1}}{1-x} = \frac{x^3 y}{(1-x)(1-x^2 y)}$$

and the other of which sums to  $xy^3/[(1-y)(1-xy^2)]$ . Therefore

$$\begin{aligned} S(x, y) &= \frac{xy}{(1-x)(1-y)} - \frac{x^3 y}{(1-x)(1-x^2 y)} \\ &\quad - \frac{xy^3}{(1-y)(1-xy^2)} \\ &= \frac{xy(1+x+y+xy-x^2 y^2)}{(1-x^2 y)(1-xy^2)} \end{aligned}$$

and the desired limit is

$$\lim_{(x,y) \rightarrow (1,1)} xy(1+x+y+xy-x^2 y^2) = 3.$$

B-4 (based on work by Daniel Stronger) We make repeated use of the following fact: if  $f$  is a differentiable function on all of  $\mathbb{R}$ ,  $\lim_{x \rightarrow -\infty} f(x) \geq 0$ , and  $f'(x) > 0$  for all  $x \in \mathbb{R}$ , then  $f(x) > 0$  for all  $x \in \mathbb{R}$ . (Proof: if  $f(y) < 0$  for some  $x$ , then  $f(x) < f(y)$  for all  $x < y$  since  $f' > 0$ , but then  $\lim_{x \rightarrow -\infty} f(x) \leq f(y) < 0$ .)

From the inequality  $f'''(x) \leq f(x)$  we obtain

$$f'' f'''(x) \leq f''(x)f'(x) < f''(x)f(x) + f'(x)^2$$

since  $f'(x)$  is positive. Applying the fact to the difference between the right and left sides, we get

$$\frac{1}{2}(f''(x))^2 < f(x)f'(x). \quad (1)$$

On the other hand, since  $f(x)$  and  $f'''(x)$  are both positive for all  $x$ , we have

$$2f'(x)f''(x) < 2f'(x)f''(x) + 2f(x)f'''(x).$$

Applying the fact to the difference between the sides yields

$$f'(x)^2 \leq 2f(x)f''(x). \quad (2)$$

Combining (1) and (2), we obtain

$$\begin{aligned} \frac{1}{2} \left( \frac{f'(x)^2}{2f(x)} \right)^2 &< \frac{1}{2} (f''(x))^2 \\ &< f(x)f'(x), \end{aligned}$$

or  $(f'(x))^3 < 8f(x)^3$ . We conclude  $f'(x) < 2f(x)$ , as desired.

Note: one can actually prove the result with a smaller constant in place of 2, as follows. Adding  $\frac{1}{2}f'(x)f'''(x)$  to both sides of (1) and again invoking the original bound  $f'''(x) \leq f(x)$ , we get

$$\begin{aligned} \frac{1}{2} [f'(x)f'''(x) + (f''(x))^2] &< f(x)f'(x) + \frac{1}{2}f'(x)f'''(x) \\ &\leq \frac{3}{2}f(x)f'(x). \end{aligned}$$

Applying the fact again, we get

$$\frac{1}{2}f'(x)f''(x) < \frac{3}{4}f(x)^2.$$

Multiplying both sides by  $f'(x)$  and applying the fact once more, we get

$$\frac{1}{6}(f'(x))^3 < \frac{1}{4}f(x)^3.$$

From this we deduce  $f'(x) < (3/2)^{1/3}f(x) < 2f(x)$ , as desired.

I don't know what the best constant is, except that it is not less than 1 (because  $f(x) = e^x$  satisfies the given conditions).

**B-5** We claim that the eigenvalues of  $A$  are 0 with multiplicity  $n-2$ , and  $n/2$  and  $-n/2$ , each with multiplicity 1. To prove this claim, define vectors  $v^{(m)}$ ,  $0 \leq m \leq n-1$ , componentwise by  $(v^{(m)})_k = e^{ikm\theta}$ , and note that the  $v^{(m)}$  form a basis for  $\mathbb{C}^n$ . (If we arrange the  $v^{(m)}$  into an  $n \times n$  matrix, then the determinant of this matrix is a Vandermonde product which is nonzero.) Now note that

$$\begin{aligned} (Av^{(m)})_j &= \sum_{k=1}^n \cos(j\theta + k\theta) e^{ikm\theta} \\ &= \frac{e^{ij\theta}}{2} \sum_{k=1}^n e^{ik(m+1)\theta} + \frac{e^{-ij\theta}}{2} \sum_{k=1}^n e^{ik(m-1)\theta}. \end{aligned}$$

Since  $\sum_{k=1}^n e^{ik\ell\theta} = 0$  for integer  $\ell$  unless  $n|\ell$ , we conclude that  $Av^{(m)} = 0$  for  $m = 0$  or for  $2 \leq m \leq n-1$ . In addition, we find that  $(Av^{(1)})_j = \frac{n}{2}e^{-ij\theta} = \frac{n}{2}(v^{(n-1)})_j$  and  $(Av^{(n-1)})_j = \frac{n}{2}e^{ij\theta} = \frac{n}{2}(v^{(1)})_j$ , so that  $A(v^{(1)} \pm v^{(n-1)}) = \pm \frac{n}{2}(v^{(1)} \pm v^{(n-1)})$ . Thus  $\{v^{(0)}, v^{(2)}, v^{(3)}, \dots, v^{(n-2)}, v^{(1)} + v^{(n-1)}, v^{(1)} - v^{(n-1)}\}$  is a basis for  $\mathbb{C}^n$  of eigenvectors of  $A$  with the claimed eigenvalues.

Finally, the determinant of  $I+A$  is the product of  $(1+\lambda)$  over all eigenvalues  $\lambda$  of  $A$ ; in this case,  $\det(I+A) = (1+n/2)(1-n/2) = 1-n^2/4$ .

**B-6** First solution: Choose a sequence  $p_1, p_2, \dots$  of primes as follows. Let  $p_1$  be any prime dividing an element of  $S$ . To define  $p_{j+1}$  given  $p_1, \dots, p_j$ , choose an integer  $N_j \in S$  relatively prime to  $p_1 \cdots p_j$  and let  $p_{j+1}$  be a prime divisor of  $N_j$ , or stop if no such  $N_j$  exists.

Since  $S$  is finite, the above algorithm eventually terminates in a finite sequence  $p_1, \dots, p_k$ . Let  $m$  be the smallest integer such that  $p_1 \cdots p_m$  has a divisor in  $S$ . (By the assumption on  $S$  with  $n = p_1 \cdots p_k$ ,  $m = k$  has this property, so  $m$  is well-defined.) If  $m = 1$ , then  $p_1 \in S$ , and we are done, so assume  $m \geq 2$ . Any divisor  $d$  of  $p_1 \cdots p_m$  in  $S$  must be a multiple of  $p_m$ , or else it would also be a divisor of  $p_1 \cdots p_{m-1}$ , contradicting the choice of  $m$ . But now  $\gcd(d, N_{m-1}) = p_m$ , as desired.

Second solution (from sci.math): Let  $n$  be the smallest integer such that  $\gcd(s, n) > 1$  for all  $s$  in  $n$ ; note that  $n$  obviously has no repeated prime factors. By the condition on  $S$ , there exists  $s \in S$  which divides  $n$ .

On the other hand, if  $p$  is a prime divisor of  $s$ , then by the choice of  $n$ ,  $n/p$  is relatively prime to some element  $t$  of  $S$ . Since  $n$  cannot be relatively prime to  $t$ ,  $t$  is divisible by  $p$ , but not by any other prime divisor of  $n$  (as those primes divide  $n/p$ ). Thus  $\gcd(s, t) = p$ , as desired.