A1 Let $n$ be a fixed positive integer. How many ways are there to write $n$ as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with $k$ an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: $4$, $2+2$, $1+1+2$, $1+1+1+1$.

A2 Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n}$$

$$\leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

A3 Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers $x$.

A4 Suppose that $a, b, c, A, B, C$ are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers $x$. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

A5 A Dyck $n$-path is a lattice path of $n$ upsteps $(1, 1)$ and $n$ downsteps $(1, -1)$ that starts at the origin $O$ and never dips below the $x$-axis. A return is a maximal sequence of contiguous downsteps that terminates on the $x$-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.

\[
\begin{array}{c}
O \\
\bullet \quad \bullet \\
\bullet \\
\end{array}
\]

Show that there is a one-to-one correspondence between the Dyck $n$-paths with no return of even length and the Dyck $(n-1)$-paths.

A6 For a set $S$ of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs $(s_1, s_2)$ such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets $A$ and $B$ in such a way that $r_S(n) = r_B(n)$ for all $n$?

B1 Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?