The 64th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2003

A–1 Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

A–2 Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$\begin{aligned}&(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \\ &\le [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.\end{aligned}$$

A–3 Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

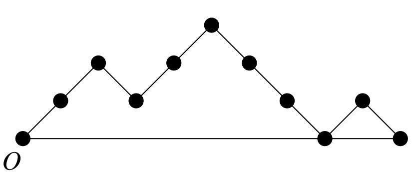
A–4 Suppose that a, b, c, A, B, C are real numbers, $a \ne 0$ and $A \ne 0$, such that

$$|ax^2 + bx + c| \le |Ax^2 + Bx + C|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \le |B^2 - 4AC|.$$

A–5 A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

A–6 For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \ne s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?

B–1 Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

B–2 Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < 2/n$.

B–3 Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \mathrm{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\le x$.)

B–4 Let

$$\begin{aligned} f(z) &= az^4 + bz^3 + cz^2 + dz + e \\ &= a(z-r_1)(z-r_2)(z-r_3)(z-r_4)\end{aligned}$$

where a, b, c, d, e are integers, $a \ne 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \ne r_3 + r_4$, then $r_1 r_2$ is a rational number.

B–5 Let A, B, and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distance from P to A, B, C, respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

B–6 Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \ge \int_0^1 |f(x)|\, dx.$$