Solutions to the 68th William Lowell Putnam Mathematical Competition
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A–1 The only such α are $2/3, 3/2, (13 \pm \sqrt{601})/12$.

First solution: Let C_1 and C_2 be the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$, respectively, and let L be the line $y = x$. We consider three cases.

If C_1 is tangent to L, then the point of tangency (x,x) satisfies

$$2\alpha x + \alpha = 1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

by symmetry, C_2 is tangent to L there, so C_1 and C_2 are tangent. Writing $\alpha = 1/(2x+1)$ in the first equation and substituting into the second, we must have

$$x = \frac{x^2+x}{2x+1} + \frac{1}{24},$$

which simplifies to $0 = 24x^2 - 2x - 1 = (6x+1)(4x-1)$, or $x \in \{1/4, -1/6\}$. This yields $\alpha = 1/(2x+1) \in \{2/3, 3/2\}$.

If C_1 does not intersect L, then C_1 and C_2 are separated by L and so cannot be tangent.

If C_1 intersects L in two distinct points P_1, P_2, then it is not tangent to L at either point. Suppose at one of these points, say P_1, the tangent to C_1 is perpendicular to L; then by symmetry, the same will be true of C_2, so C_1 and C_2 will be tangent at P_1. In this case, the point $P_1 = (x,x)$ satisfies

$$2\alpha x + \alpha = -1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

writing $\alpha = -1/(2x+1)$ in the first equation and substituting into the second, we have

$$x = -\frac{x^2+x}{2x+1} + \frac{1}{24},$$

or $x = (-23 \pm \sqrt{601})/72$. This yields $\alpha = -1/(2x+1) = (13 \pm \sqrt{601})/12$.

If instead the tangents to C_1 at P_1, P_2 are not perpendicular to L, then we claim there cannot be any point where C_1 and C_2 are tangent. Indeed, if we count intersections of C_1 and C_2 (by using C_1 to substitute for y in C_2, then solving for y), we get at most four solutions counting multiplicity. Two of these are P_1 and P_2, and any point of tangency counts for two more. However, off of L, any point of tangency would have a mirror image which is also a point of tangency, and there cannot be six solutions. Hence we have now found all possible α.

Second solution: For any nonzero value of α, the two conics will intersect in four points in the complex projective plane $\mathbb{P}^2(\mathbb{C})$. To determine the y-coordinates of these intersection points, subtract the two equations to obtain

$$(y-x) = \alpha(x-y)(x+y) + \alpha(x-y).$$

Therefore, at a point of intersection we have either $x = y$, or $x = -1/\alpha - (y+1)$. Substituting these two possible linear conditions into the second equation shows that the y-coordinate of a point of intersection is a root of either $Q_1(y) = \alpha y^2 + (\alpha-1)y + 1/24$ or $Q_2(y) = \alpha y^2 + (\alpha+1)y + 25/24 + 1/\alpha$.

If two curves are tangent, then the y-coordinates of at least two of the intersection points will coincide; the converse is also true because one of the curves is the graph of a function in x. The coincidence occurs precisely when either the discriminant of at least one of Q_1 or Q_2 is zero, or there is a common root of Q_1 and Q_2. Computing the discriminants of Q_1 and Q_2 yields (up to constant factors) $f_1(\alpha) = 6\alpha^2 - 13\alpha + 6$ and $f_2(\alpha) = 6\alpha^2 - 13\alpha - 18$, respectively. If on the other hand Q_1 and Q_2 have a common root, it must be also a root of $Q_2(y) - Q_1(y) = 2y + 1 + 1/\alpha$, yielding $y = -(1+\alpha)/(2\alpha)$ and $0 = Q_1(y) = -f_2(\alpha)/(24\alpha)$.

Thus the values of α for which the two curves are tangent must be contained in the set of zeros of f_1 and f_2, namely $2/3$, $3/2$, and $(13 \pm \sqrt{601})/12$.

Remark: The fact that the two conics in $\mathbb{P}^2(\mathbb{C})$ meet in four points, counted with multiplicities, is a special case of *Bézout's theorem*: two curves in $\mathbb{P}^2(\mathbb{C})$ of degrees m, n and not sharing any common component meet in exactly mn points when counted with multiplicity.

Many solvers were surprised that the proposers chose the parameter $1/24$ to give two rational roots and two nonrational roots. In fact, they had no choice in the matter: attempting to make all four roots rational by replacing $1/24$ by β amounts to asking for $\beta^2 + \beta$ and $\beta^2 + \beta + 1$ to be perfect squares. This cannot happen outside of trivial cases ($\beta = 0, -1$) ultimately because the elliptic curve 24A1 (in Cremona's notation) over $\mathbb{Q}$ has rank 0. (Thanks to Noam Elkies for providing this computation.)

However, there are choices that make the radical milder, e.g., $\beta = 1/3$ gives $\beta^2 + \beta = 4/9$ and $\beta^2 + \beta + 1 = 13/9$, while $\beta = 3/5$ gives $\beta^2 + \beta = 24/25$ and $\beta^2 + \beta + 1 = 49/25$.

A–2 The minimum is 4, achieved by the square with vertices $(\pm 1, \pm 1)$.

First solution: To prove that 4 is a lower bound, let S be a convex set of the desired form. Choose $A,B,C,D \in S$ lying on the branches of the two hyperbolas, with A in the upper right quadrant, B in the upper left, C in the lower left, D in the lower right. Then the area of the quadrilateral $ABCD$ is a lower bound for the area of S.

Write $A = (a,1/a)$, $B = (b,-1/b)$, $C = (-c,-1/c)$, $D = (-d,1/d)$ with $a,b,c,d > 0$. Then the area of the quadrilateral $ABCD$ is

$$\frac{1}{2}(a/b+b/c+c/d+d/a+b/a+c/b+d/c+a/d),$$

which by the arithmetic-geometric mean inequality is at least 4.

Second solution: Choose A,B,C,D as in the first solution. Note that both the hyperbolas and the area of the convex hull of $ABCD$ are invariant under the transformation $(x,y) \mapsto (xm,y/m)$ for any $m > 0$. For m small, the counterclockwise angle from the line AC to the line BD approaches 0; for m large, this angle approaches π. By continuity, for some m this angle becomes $\pi/2$, that is, AC and BD become perpendicular. The area of $ABCD$ is then $AC \cdot BD$.

It thus suffices to note that $AC \geq 2\sqrt{2}$ (and similarly for BD). This holds because if we draw the tangent lines to the hyperbola $xy = 1$ at the points $(1,1)$ and $(-1,-1)$, then A and C lie outside the region between these lines. If we project the segment AC orthogonally onto the line $x = y = 1$, the resulting projection has length at least $2\sqrt{2}$, so AC must as well.

Third solution: (by Richard Stanley) Choose A,B,C,D as in the first solution. Now fixing A and C, move B and D to the points at which the tangents to the curve are parallel to the line AC. This does not increase the area of the quadrilateral $ABCD$ (even if this quadrilateral is not convex).

Note that B and D are now diametrically opposite; write $B = (-x,1/x)$ and $D = (x,-1/x)$. If we thus repeat the procedure, fixing B and D and moving A and C to the points where the tangents are parallel to BD, then A and C must move to $(x,1/x)$ and $(-x,-1/x)$, respectively, forming a rectangle of area 4.

Remark: Many geometric solutions are possible. An example suggested by David Savitt (due to Chris Brewer): note that AD and BC cross the positive and negative x-axes, respectively, so the convex hull of $ABCD$ contains O. Then check that the area of triangle OAB is at least 1, et cetera.

A–3 Assume that we have an ordering of $1,2,\dots,3k+1$ such that no initial subsequence sums to 0 mod 3. If we omit the multiples of 3 from this ordering, then the remaining sequence mod 3 must look like $1,1,-1,1,-1,\dots$ or $-1,-1,1,-1,1,\dots$. Since there is one more integer in the ordering congruent to 1 mod 3 than to -1, the sequence mod 3 must look like $1,1,-1,1,-1,\dots$.

It follows that the ordering satisfies the given condition if and only if the following two conditions hold: the first element in the ordering is not divisible by 3, and the sequence mod 3 (ignoring zeroes) is of the form $1,1,-1,1,-1,\dots$. The two conditions are independent, and the probability of the first is $(2k+1)/(3k+1)$ while the probability of the second is $1/\binom{2k+1}{k}$, since there are $\binom{2k+1}{k}$ ways to order $(k+1)$ 1's and k -1's. Hence the desired probability is the product of these two, or $\frac{k!(k+1)!}{(3k+1)(2k)!}$.

A–4 Note that n is a repunit if and only if $9n+1 = 10^m$ for some power of 10 greater than 1. Consequently, if we put

$$g(n) = 9f\left(\frac{n-1}{9}\right)+1,$$

then f takes repunits to repunits if and only if g takes powers of 10 greater than 1 to powers of 10 greater than 1. We will show that the only such functions g are those of the form $g(n) = 10^c n^d$ for $d \geq 0$, $c \geq 1-d$ (all of which clearly work), which will mean that the desired polynomials f are those of the form

$$f(n) = \frac{1}{9}(10^c(9n+1)^d - 1)$$

for the same c,d.

It is convenient to allow "powers of 10" to be of the form 10^k for any integer k. With this convention, it suffices to check that the polynomials g taking powers of 10 greater than 1 to powers of 10 are of the form $10^c n^d$ for any integers c,d with $d \geq 0$.

First solution: Suppose that the leading term of $g(x)$ is ax^d, and note that $a > 0$. As $x \to \infty$, we have $g(x)/x^d \to a$; however, for x a power of 10 greater than 1, $g(x)/x^d$ is a power of 10. The set of powers of 10 has no positive limit point, so $g(x)/x^d$ must be equal to a for $x = 10^k$ with k sufficiently large, and we must have $a = 10^c$ for some c. The polynomial $g(x) - 10^c x^d$ has infinitely many roots, so must be identically zero.

Second solution: We proceed by induction on $d = \deg(g)$. If $d = 0$, we have $g(n) = 10^c$ for some c. Otherwise, g has rational coefficients by Lagrange's interpolation formula (this applies to any polynomial of degree d taking at least $d+1$ different rational numbers to rational numbers), so $g(0) = t$ is rational. Moreover, g takes each value only finitely many times, so the sequence $g(10^0), g(10^1), \dots$ includes arbitrarily large powers of 10. Suppose that $t \neq 0$; then we can choose a positive integer h such that the numerator of t is not divisible by 10^h. But for c large enough, $g(10^c) - t$ has numerator divisible by 10^b for some $b > h$, contradiction.

Consequently, $t = 0$, and we may apply the induction hypothesis to $g(n)/n$ to deduce the claim.

Remark: The second solution amounts to the fact that g, being a polynomial with rational coefficients, is continuous for the 2-adic and 5-adic topologies on $\mathbb{Q}$. By contrast, the first solution uses the "∞-adic" topology, i.e., the usual real topology.

A–5 In all solutions, let G be a finite group of order m.

First solution: By Lagrange's theorem, if m is not divisible by p, then $n = 0$. Otherwise, let S be the set of p-tuples $(a_0, \dots, a_{p-1}) \in G^p$ such that $a_0 \cdots a_{p-1} = e$; then S has cardinality m^{p-1}, which is divisible by p. Note that this set is invariant under cyclic permutation, that is, if $(a_0, \dots, a_{p-1}) \in S$, then $(a_1, \dots, a_{p-1}, a_0) \in S$ also. The fixed points under this operation are the tuples $(a, \dots, a)$ with $a^p = e$; all other tuples can be grouped into orbits under cyclic permutation, each of which has size p. Consequently, the number of $a \in G$ with $a^p = e$ is divisible by p; since that number is $n+1$ (only e has order 1), this proves the claim.

Second solution: (by Anand Deopurkar) Assume that $n > 0$, and let H be any subgroup of G of order p. Let S be the set of all elements of $G \setminus H$ of order dividing p, and let H act on G by conjugation. Each orbit has size p except for those which consist of individual elements g which commute with H. For each such g, g and H generate an elementary abelian subgroup of G of order p^2. However, we can group these g into sets of size $p^2 - p$ based on which subgroup they generate together with H. Hence the cardinality of S is divisible by p; adding the $p-1$ nontrivial elements of H gives $n \equiv -1 \pmod{p}$ as desired.

Third solution: Let S be the set of elements in G having order dividing p, and let H be an elementary abelian p-group of maximal order in G. If $|H| = 1$, then we are done. So assume $|H| = p^k$ for some $k \geq 1$, and let H act on S by conjugation. Let $T \subset S$ denote the set of fixed points of this action. Then the size of every H-orbit on S divides p^k, and so $|S| \equiv |T| \pmod{p}$. On the other hand, $H \subset T$, and if T contained an element not in H, then that would contradict the maximality of H. It follows that $H = T$, and so $|S| \equiv |T| = |H| = p^k \equiv 0 \pmod{p}$, i.e., $|S| = n+1$ is a multiple of p.

Remark: This result is a theorem of Cauchy; the first solution above is due to McKay. A more general (and more difficult) result was proved by Frobenius: for any positive integer m, if G is a finite group of order divisible by m, then the number of elements of G of order dividing m is a multiple of m.

A–6 For an admissible triangulation $\mathscr{T}$, number the vertices of P consecutively $v_1, \dots, v_n$, and let a_i be the number of edges in $\mathscr{T}$ emanating from v_i; note that $a_i \geq 2$ for all i.

We first claim that $a_1 + \cdots + a_n \leq 4n - 6$. Let V, E, F denote the number of vertices, edges, and faces in $\mathscr{T}$. By Euler's Formula, $(F+1) - E + V = 2$ (one must add 1 to the face count for the region exterior to P). Each face has three edges, and each edge but the n outside edges belongs to two faces; hence $F = 2E - n$. On the other hand, each edge has two endpoints, and each of the $V - n$ internal vertices is an endpoint of at least 6 edges; hence $a_1 + \cdots + a_n + 6(V - n) \leq 2E$. Combining this inequality with the previous two equations gives

$$\begin{aligned} a_1 + \cdots + a_n &\leq 2E + 6n - 6(1 - F + E) \\ &= 4n - 6, \end{aligned}$$

as claimed.

Now set $A_3 = 1$ and $A_n = A_{n-1} + 2n - 3$ for $n \geq 4$; we will prove by induction on n that $\mathscr{T}$ has at most A_n triangles. For $n = 3$, since $a_1 + a_2 + a_3 = 6$, $a_1 = a_2 = a_3 = 2$ and hence $\mathscr{T}$ consists of just one triangle.

Next assume that an admissible triangulation of an $(n-1)$-gon has at most A_{n-1} triangles, and let $\mathscr{T}$ be an admissible triangulation of an n-gon. If any $a_i = 2$, then we can remove the triangle of $\mathscr{T}$ containing vertex v_i to obtain an admissible triangulation of an $(n-1)$-gon; then the number of triangles in $\mathscr{T}$ is at most $A_{n-1} + 1 < A_n$ by induction. Otherwise, all $a_i \geq 3$. Now the average of $a_1, \dots, a_n$ is less than 4, and thus there are more $a_i = 3$ than $a_i \geq 5$. It follows that there is a sequence of k consecutive vertices in P whose degrees are $3, 4, 4, \dots, 4, 3$ in order, for some k with $2 \leq k \leq n-1$ (possibly $k = 2$, in which case there are no degree 4 vertices separating the degree 3 vertices). If we remove from $\mathscr{T}$ the $2k-1$ triangles which contain at least one of these vertices, then we are left with an admissible triangulation of an $(n-1)$-gon. It follows that there are at most $A_{n-1} + 2k - 1 \leq A_{n-1} + 2n - 3 = A_n$ triangles in $\mathscr{T}$. This completes the induction step and the proof.

Remark: We can refine the bound A_n somewhat. Supposing that $a_i \geq 3$ for all i, the fact that $a_1 + \cdots + a_n \leq 4n - 6$ implies that there are at least six more indices i with $a_i = 3$ than with $a_i \geq 5$. Thus there exist six sequences with degrees $3, 4, \dots, 4, 3$, of total length at most $n + 6$. We may thus choose a sequence of length $k \leq \lfloor \frac{n}{6} \rfloor + 1$, so we may improve the upper bound to $A_n = A_{n-1} + 2\lfloor \frac{n}{6} \rfloor + 1$, or asymptotically $\frac{1}{6}n^2$.

However (as noted by Noam Elkies), a hexagonal swatch of a triangular lattice, with the boundary as close to regular as possible, achieves asymptotically $\frac{1}{6}n^2$ triangles.

B–1 The problem fails if f is allowed to be constant, e.g., take $f(n) = 1$. We thus assume that f is nonconstant. Write $f(n) = \sum_{i=0}^d a_i n^i$ with $a_i > 0$. Then

$$\begin{aligned} f(f(n)+1) &= \sum_{i=0}^d a_i (f(n)+1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If $n = 1$, then this implies that $f(f(n)+1)$ is divisible by $f(n)$. Otherwise, $0 < f(1) < f(n)$ since f is nonconstant and has positive coefficients, so $f(f(n)+1)$ cannot be divisible by $f(n)$.

B–2 Put $B = \max_{0 \leq x \leq 1} |f'(x)|$ and $g(x) = \int_0^x f(y)\,dy$. Since $g(0) = g(1) = 0$, the maximum value of $|g(x)|$ must occur at a critical point $y \in (0,1)$ satisfying $g'(y) = f(y) = 0$. We may thus take $\alpha = y$ hereafter.

Since $\int_0^\alpha f(x)\,dx = -\int_0^{1-\alpha} f(1-x)\,dx$, we may assume that $\alpha \leq 1/2$. By then substituting $-f(x)$ for $f(x)$ if needed, we may assume that $\int_0^\alpha f(x)\,dx \geq 0$. From the inequality $f'(x) \geq -B$, we deduce $f(x) \leq B(\alpha - x)$ for $0 \leq x \leq \alpha$, so

$$\begin{aligned}\int_0^\alpha f(x)\,dx &\leq \int_0^\alpha B(\alpha - x)\,dx \\ &= -\frac{1}{2}B(\alpha-x)^2 \Big|_0^\alpha \\ &= \frac{\alpha^2}{2}B \leq \frac{1}{8}B\end{aligned}$$

as desired.

B–3 **First solution:** Observing that $x_2/2 = 13$, $x_3/4 = 34$, $x_4/8 = 89$, we guess that $x_n = 2^{n-1}F_{2n+3}$, where F_k is the k-th Fibonacci number. Thus we claim that $x_n = \frac{2^{n-1}}{\sqrt{5}}(\alpha^{2n+3} - \alpha^{-(2n+3)})$, where $\alpha = \frac{1+\sqrt{5}}{2}$, to make the answer $x_{2007} = \frac{2^{2006}}{\sqrt{5}}(\alpha^{3997} - \alpha^{-3997})$.

We prove the claim by induction; the base case $x_0 = 1$ is true, and so it suffices to show that the recursion $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$ is satisfied for our formula for x_n. Indeed, since $\alpha^2 = \frac{3+\sqrt{5}}{2}$, we have

$$\begin{aligned}x_{n+1} - (3+\sqrt{5})x_n &= \frac{2^{n-1}}{\sqrt{5}}(2(\alpha^{2n+5} - \alpha^{-(2n+5)}) \\ &\quad - (3+\sqrt{5})(\alpha^{2n+3} - \alpha^{-(2n+3)})) \\ &= 2^n\alpha^{-(2n+3)}.\end{aligned}$$

Now $2^n\alpha^{-(2n+3)} = (\frac{1-\sqrt{5}}{2})^3(3-\sqrt{5})^n$ is between -1 and 0; the recursion follows since x_n, x_{n+1} are integers.

Second solution: (by Catalin Zara) Since x_n is rational, we have $0 < x_n\sqrt{5} - \lfloor x_n\sqrt{5} \rfloor < 1$. We now have the inequalities

$$\begin{aligned}x_{n+1} - 3x_n < x_n\sqrt{5} &< x_{n+1} - 3x_n + 1 \\ (3+\sqrt{5})x_n - 1 < x_{n+1} &< (3+\sqrt{5})x_n \\ 4x_n - (3-\sqrt{5}) < (3-\sqrt{5})x_{n+1} &< 4x_n \\ 3x_{n+1} - 4x_n < x_{n+1}\sqrt{5} &< 3x_{n+1} - 4x_n + (3-\sqrt{5}).\end{aligned}$$

Since $0 < 3-\sqrt{5} < 1$, this yields $\lfloor x_{n+1}\sqrt{5} \rfloor = 3x_{n+1} - 4x_n$, so we can rewrite the recursion as $x_{n+1} = 6x_n - 4x_{n-1}$ for $n \geq 2$. It is routine to solve this recursion to obtain the same solution as above.

Remark: With an initial 1 prepended, this becomes sequence A018903 in Sloane's On-Line Encyclopedia of Integer Sequences: (`http://www.research.att.com/~njas/sequences/`). Therein, the sequence is described as the case $S(1,5)$ of the sequence $S(a_0,a_1)$ in which a_{n+2} is the least integer for which $a_{n+2}/a_{n+1} > a_{n+1}/a_n$. Sloane cites D. W. Boyd, Linear recurrence relations for some generalized Pisot sequences, *Advances in Number Theory (Kingston, ON, 1991)*, Oxford Univ. Press, New York, 1993, p. 333–340.

B–4 The number of pairs is 2^{n+1}. The degree condition forces P to have degree n and leading coefficient ± 1; we may count pairs in which P has leading coefficient 1 as long as we multiply by 2 afterward.

Factor both sides:

$$\begin{aligned}&(P(X) + Q(X)i)(P(X) - Q(X)i) \\ &= \prod_{j=0}^{n-1}(X - \exp(2\pi i(2j+1)/(4n))) \\ &\quad \cdot \prod_{j=0}^{n-1}(X + \exp(2\pi i(2j+1)/(4n))).\end{aligned}$$

Then each choice of P,Q corresponds to equating $P(X) + Q(X)i$ with the product of some n factors on the right, in which we choose exactly of the two factors for each $j = 0, \dots, n-1$. (We must take exactly n factors because as a polynomial in X with complex coefficients, $P(X) + Q(X)i$ has degree exactly n. We must choose one for each j to ensure that $P(X) + Q(X)i$ and $P(X) - Q(X)i$ are complex conjugates, so that P,Q have real coefficients.) Thus there are 2^n such pairs; multiplying by 2 to allow P to have leading coefficient -1 yields the desired result.

Remark: If we allow P and Q to have complex coefficients but still require $\deg(P) > \deg(Q)$, then the number of pairs increases to $2\binom{2n}{n}$, as we may choose any n of the $2n$ factors of $X^{2n} + 1$ to use to form $P(X) + Q(X)i$.

B–5 For n an integer, we have $\left\lfloor \frac{n}{k} \right\rfloor = \frac{n-j}{k}$ for j the unique integer in $\{0, \dots, k-1\}$ congruent to n modulo k; hence

$$\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n}{k} \right\rfloor - \frac{n-j}{k}\right) = 0.$$

By expanding this out, we obtain the desired polynomials $P_0(n), \dots, P_{k-1}(n)$.

Remark: Variants of this solution are possible that construct the P_i less explicitly, using Lagrange interpolation or Vandermonde determinants.

B–6 (Suggested by Oleg Golberg) Assume $n \geq 2$, or else the problem is trivially false. Throughout this proof, any C_i will be a positive constant whose exact value is immaterial. As in the proof of Stirling's approximation, we estimate for any fixed $c \in \mathbb{R}$,

$$\sum_{i=1}^n (i+c)\log i = \frac{1}{2}n^2\log n - \frac{1}{4}n^2 + O(n\log n)$$

by comparing the sum to an integral. This gives

$$\begin{aligned} n^{n^2/2-C_1 n} e^{-n^2/4} &\le 1^{1+c} 2^{2+c} \cdots n^{n+c} \\ &\le n^{n^2/2+C_2 n} e^{-n^2/4}. \end{aligned}$$

We now interpret $f(n)$ as counting the number of n-tuples $(a_1, \ldots, a_n)$ of nonnegative integers such that

$$a_1 1! + \cdots + a_n n! = n!.$$

For an upper bound on $f(n)$, we use the inequalities $0 \le a_i \le n!/i!$ to deduce that there are at most $n!/i! + 1 \le 2(n!/i!)$ choices for a_i. Hence

$$\begin{aligned} f(n) &\le 2^n \frac{n!}{1!} \cdots \frac{n!}{n!} \\ &= 2^n 2^1 3^2 \cdots n^{n-1} \\ &\le n^{n^2/2+C_3 n} e^{-n^2/4}. \end{aligned}$$

For a lower bound on $f(n)$, we note that if $0 \le a_i < (n-1)!/i!$ for $i = 2, \ldots, n-1$ and $a_n = 0$, then $0 \le a_2 2! + \cdots + a_n n! \le n!$, so there is a unique choice of a_1 to complete this to a solution of $a_1 1! + \cdots + a_n n! = n!$. Hence

$$\begin{aligned} f(n) &\ge \frac{(n-1)!}{2!} \cdots \frac{(n-1)!}{(n-1)!} \\ &= 3^1 4^2 \cdots (n-1)^{n-3} \\ &\ge n^{n^2/2+C_4 n} e^{-n^2/4}. \end{aligned}$$