The 69th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2008

A–1 Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function such that $f(x,y)+f(y,z)+f(z,x)=0$ for all real numbers x, y, and z. Prove that there exists a function $g:\mathbb{R}\to\mathbb{R}$ such that $f(x,y)=g(x)-g(y)$ for all real numbers x and y.

A–2 Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

A–3 Start with a finite sequence $a_1,a_2,\dots,a_n$ of positive integers. If possible, choose two indices $j<k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j,a_k)$ and $\operatorname{lcm}(a_j,a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

A–4 Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases} x & \text{if } x\le e \\ xf(\ln x) & \text{if } x>e. \end{cases}$$

Does $\sum_{n=1}^{\infty}\frac{1}{f(n)}$ converge?

A–5 Let $n\ge 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1),g(1)),(f(2),g(2)),\dots,(f(n),g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

A–6 Prove that there exists a constant $c>0$ such that in every nontrivial finite group G there exists a sequence of length at most $c\ln|G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4,4,2$ is a subsequence of $2,4,6,4,2$, but $2,2,4$ is not.)

B–1 What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

B–2 Let $F_0(x)=\ln x$. For $n\ge 0$ and $x>0$, let $F_{n+1}(x)=\int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n\to\infty}\frac{n!F_n(1)}{\ln n}.$$

B–3 What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

B–4 Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0),h(1),\dots,h(p^2-1)$ are distinct modulo p^2. Show that $h(0),h(1),\dots,h(p^3-1)$ are distinct modulo p^3.

B–5 Find all continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q=a/b$ for some integer a with $\gcd(a,b)=1$.) (Note: gcd means greatest common divisor.)

B–6 Let n and k be positive integers. Say that a permutation σ of $\{1,2,\dots,n\}$ is *k-limited* if $|\sigma(i)-i|\le k$ for all i. Prove that the number of k-limited permutations of $\{1,2,\dots,n\}$ is odd if and only if $n\equiv 0$ or $1 \pmod{2k+1}$.