The 72nd William Lowell Putnam Mathematical Competition
Saturday, December 3, 2011

A–1 Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0,0), P_1, \dots, P_n$ such that $n \geq 2$ and:

- The directed line segments $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.
- The lengths of these line segments are positive and strictly increasing.

[Picture omitted.] How many of the points (x,y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n of any growing spiral?

A–2 Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \dots a_n}$$

converges, and evaluate S.

A–3 Find a real number c and a positive number L for which

$$\lim_{r \to \infty} \frac{r^c \int_0^{\pi/2} x^r \sin x \, dx}{\int_0^{\pi/2} x^r \cos x \, dx} = L.$$

A–4 For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

A–5 Let $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable functions with the following properties:

- $F(u,u) = 0$ for every $u \in \mathbb{R}$;
- for every $x \in \mathbb{R}$, $g(x) > 0$ and $x^2 g(x) \leq 1$;
- for every $(u,v) \in \mathbb{R}^2$, the vector $\nabla F(u,v)$ is either $\mathbf{0}$ or parallel to the vector $\langle g(u), -g(v) \rangle$.

Prove that there exists a constant C such that for every $n \geq 2$ and any $x_1, \dots, x_{n+1} \in \mathbb{R}$, we have

$$\min_{i \neq j} |F(x_i, x_j)| \leq \frac{C}{n}.$$

A–6 Let G be an abelian group with n elements, and let

$$\{g_1 = e, g_2, \dots, g_k\} \subsetneq G$$

be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, ..., g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$. Prove that there exists a real number $b \in (0,1)$ such that

$$\lim_{m \to \infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\text{Prob}(g = x) - \frac{1}{n} \right)^2$$

is positive and finite.

B–1 Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon.$$

B–2 Let S be the set of all ordered triples (p,q,r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

B–3 Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

B–4 In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h,k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

B–5 Let $a_1, a_2, \dots$ be real numbers. Suppose that there is a constant A such that for all n,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1 + (x - a_i)^2} \right)^2 dx \leq An.$$

Prove there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} (1 + (a_i - a_j)^2) \geq Bn^3.$$

B–6 Let p be an odd prime. Show that for at least $(p+1)/2$ values of n in $\{0, 1, 2, \dots, p-1\}$, $\sum_{k=0}^{p-1} k! n^k$ is not divisible by p.