A1 Define a growing spiral in the plane to be a sequence of points with integer coordinates \( P_0 = (0,0), P_1, \ldots, P_n \) such that \( n \geq 2 \) and:

- the directed line segments \( P_0 P_1, P_1 P_2, \ldots, P_{n-1} P_n \)
  are in the successive coordinate directions east (for \( P_0 P_1 \)), north, west, south, east, etc.:

- the lengths of these line segments are positive and strictly increasing.

[Picture omitted.] How many of the points \((x,y)\) with integer coordinates \(0 \leq x \leq 2011,0 \leq y \leq 2011\) cannot be the last point, \(P_n\) of any growing spiral?

A2 Let \(a_1,a_2,\ldots\) and \(b_1,b_2,\ldots\) be sequences of positive real numbers such that \(a_1 = b_1 = 1\) and \(b_n = b_{n-1}a_n - 2\) for \(n = 2,3,\ldots\). Assume that the sequence \((b_j)\) is bounded. Prove that

\[
S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}
\]
converges, and evaluate \(S\).

A3 Find a real number \(c\) and a positive number \(L\) for which

\[
\lim_{r \to \infty} \int_0^{\pi/2} r^2 x^2 \sin x dx = L.
\]

A4 For which positive integers \(n\) is there an \(n \times n\) matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

A5 Let \(F : \mathbb{R}^2 \to \mathbb{R}\) and \(g : \mathbb{R} \to \mathbb{R}\) be twice continuously differentiable functions with the following properties:

- \(F(u,u) = 0\) for every \(u \in \mathbb{R}\);
- for every \(x \in \mathbb{R}\), \(g(x) > 0\) and \(x^2 g(x) \leq 1\);
- for every \((u,v) \in \mathbb{R}^2\), the vector \(\nabla F(u,v)\) is either \(0\) or parallel to the vector \(\langle g(u), -g(v) \rangle\).

Prove that there exists a constant \(C\) such that for every \(n \geq 2\) and any \(x_1, \ldots, x_{n+1} \in \mathbb{R}\), we have

\[
\min_{i \neq j} |F(x_i,x_j)| \leq \frac{C}{n}.
\]

A6 Let \(G\) be an abelian group with \(n\) elements, and let

\[
\{g_1 = e, g_2, \ldots, g_k\} \subseteq G
\]
be a (not necessarily minimal) set of distinct generators of \(G\). A special die, which randomly selects one of the elements \(g_1, g_2, \ldots, g_k\) with equal probability, is rolled \(n\) times and the selected elements are multiplied to produce an element \(g \in G\). Prove that there exists a real number \(b \in (0,1)\) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{x \in \mathbb{R}} \left( \text{Prob}(g = x) - \frac{1}{n} \right)^2 = 0.
\]

is positive and finite.

B1 Let \(h\) and \(k\) be positive integers. Prove that for every \(\varepsilon > 0\), there are positive integers \(m\) and \(n\) such that

\[
\varepsilon < |h \sqrt{m} - k \sqrt{n}| < 2\varepsilon.
\]

B2 Let \(S\) be the set of all ordered triples \((p,q,r)\) of prime numbers for which at least one rational number \(x\) satisfies \(px^2 + qx + r = 0\). Which primes appear in seven or more elements of \(S\)?

B3 Let \(f\) and \(g\) be (real-valued) functions defined on an open interval containing 0, with \(g\) nonzero and continuous at 0. If \(f g\) and \(f/g\) are differentiable at 0, must \(f\) be differentiable at 0? (Solution in Appendix)

B4 In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011 matrices, \(T\) and \(W\). Initially, \(T = W = 0\). After every game, for every \((h,k)\) (including for \(h = k\)), if players \(h\) and \(k\) tied (that is, both won or both lost), the entry \(T_{hk}\) is increased by 1, while if player \(h\) won and player \(k\) lost, the entry \(W_{hk}\) is increased by 1 and \(W_{kh}\) is decreased by 1.

Prove that at the end of the tournament, \(\det(T + iW)\) is a non-negative integer divisible by 2^{2010}.

B5 Let \(a_1,a_2,\ldots\) be real numbers. Suppose that there is a constant \(A\) such that for all \(n\),

\[
\int_{-\infty}^{\infty} \left( \sum_{i=1}^{n} \frac{1}{1 + (x-a_i)^2} \right)^2 dx \leq An.
\]

Prove there is a constant \(B > 0\) such that for all \(n\),

\[
\sum_{i,j=1}^{n} (1 + (a_i-a_j)^2) \geq Bn^3.
\]

B6 Let \(p\) be an odd prime. Show that for at least \((p+1)/2\) values of \(n\) in \(\{0,1,2,\ldots,p-1\}\),

\[
\sum_{k=0}^{p-1} k^n b^k
\]
is not divisible by \(p\).