The 74th William Lowell Putnam Mathematical Competition
Saturday, December 7, 2013

A1 Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

A2 Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

A3 Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

A4 A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \geq 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that $|Z(w) - Z(w')| \leq 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k} \sum_{j=1}^{k} Z(w_j) \text{ and } N = \frac{1}{k} \sum_{j=1}^{k} N(w_j)$$

are both integers. Prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.

A5 For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} ($1 \leq i << j < k \leq m$) is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \mathrm{Area}(\Delta A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

A6 Define a function $w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a,b)$ be as in the table shown; otherwise, let $w(a,b) = 0$.

$w(a,b)$		b				
		-2	-1	0	1	2
	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
a	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite subset S of $\mathbb{Z} \times \mathbb{Z}$, define

$$A(S) = \sum_{(\mathbf{s},\mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}').$$

Prove that if S is any finite nonempty subset of $\mathbb{Z} \times \mathbb{Z}$, then $A(S) > 0$. (For example, if $S = \{(0,1),(0,2),(2,0),(3,1)\}$, then the terms in $A(S)$ are $12, 12, 12, 12, 4, 4, 0, 0, 0, 0, -1, -1, -2, -2, -4, -4$.)

B1 For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

B2 Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

B3 Let P be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in P$, then $S \cup S' \in P$ and $S \cap S' \in P$, and

(ii) if $S \in P$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in P$ and T contains exactly one fewer element than S.

Suppose that $f : P \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in P.$$

Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in P$?

B4 For any continuous real-valued function f defined on the interval $[0,1]$, let

$$\mu(f) = \int_0^1 f(x)\,dx, \ \mathrm{Var}(f) = \int_0^1 (f(x) - \mu(f))^2\,dx,$$
$$M(f) = \max_{0 \le x \le 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0,1]$, then

$$\mathrm{Var}(fg) \le 2\mathrm{Var}(f)M(g)^2 + 2\mathrm{Var}(g)M(f)^2.$$

B5 Let $X = \{1, 2, \dots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \ge 0$ such that $f^{(j)}(x) \le k$. [Here $f^{(j)}$ denotes the j^{th} iterate of f, so that $f^{(0)}(x) = x$ and $f^{(j+1)}(x) = f(f^{(j)}(x))$.]

B6 Let $n \ge 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?