A1 Suppose \( X \) is a random variable that takes on only non-negative integer values, with \( E[X] = 1 \), \( E[X^2] = 2 \), and \( E[X^3] = 5 \). (Here \( E[y] \) denotes the expectation of the random variable \( Y \).) Determine the smallest possible value of the probability of the event \( X = 0 \).

A2 Let \( A \) be the \( n \times n \) matrix whose entry in the \( i \)-th row and \( j \)-th column is

\[
\frac{1}{\min(i,j)}
\]

for \( 1 \leq i, j \leq n \). Compute \( \det(A) \).

A3 Let \( a_0 = 5/2 \) and \( a_k = a_{k-1}^2 - 2 \) for \( k \geq 1 \). Compute

\[
\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)
\]

in closed form.

A4 Suppose \( X \) is a random variable that takes on only non-negative integer values, with \( E[X] = 1 \), \( E[X^2] = 2 \), and \( E[X^3] = 5 \). (Here \( E[y] \) denotes the expectation of the random variable \( Y \).) Determine the smallest possible value of the probability of the event \( X = 0 \).

A5 Let

\[
P_n(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}.
\]

Prove that the polynomials \( P_j(x) \) and \( P_k(x) \) are relatively prime for all positive integers \( j \) and \( k \) with \( j \neq k \).

A6 Let \( n \) be a positive integer. What is the largest \( k \) for which there exist \( n \times n \) matrices \( M_1, \ldots, M_k \) and \( N_1, \ldots, N_k \) with real entries such that for all \( i \) and \( j \), the matrix product \( M_i N_j \) has a zero entry somewhere on its diagonal if and only if \( i \neq j \)?

B1 A base 10 over-expansion of a positive integer \( N \) is an expression of the form

\[
N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0
\]

with \( d_k \neq 0 \) and \( d_i \in \{0, 1, 2, \ldots, 10\} \) for all \( i \). For instance, the integer \( N = 10 \) has two base 10 over-expansions: \( 10 = 10 \cdot 10^0 \) and the usual base 10 expansion \( 10 = 1 \cdot 10^1 + 0 \cdot 10^0 \). Which positive integers have a unique base 10 over-expansion?

B2 Suppose that \( f \) is a function on the interval \([1, 3]\) such that \(-1 \leq f(x) \leq 1\) for all \( x \) and \( \int_1^3 f(x) \, dx = 0 \). How large can \( \int_1^3 \frac{f(y)}{y} \, dy \) be?

B3 Let \( A \) be an \( m \times n \) matrix with rational entries. Suppose that there are at least \( m + n \) distinct prime numbers among the absolute values of the entries of \( A \). Show that the rank of \( A \) is at least 2.

B4 Show that for each positive integer \( n \), all the roots of the polynomial

\[
\sum_{k=0}^{n} 2^{k(n-k)} x^k
\]

are real numbers.

B5 In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible \( n \times n \) matrices with entries in the field \( \mathbb{Z}/p\mathbb{Z} \) of integers modulo \( p \), where \( n \) is a fixed positive integer and \( p \) is a fixed prime number. The rules of the game are:

1. A player cannot choose an element that has been chosen by either player on any previous turn.
2. A player can only choose an element that commutes with all previously chosen elements.
3. A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on \( n \) and \( p \)).