The relative class number one problem for function fields, II

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These slides can be downloaded from https://kskedlaya.org/slides/.
Jupyter notebooks available from https://github.com/kedlaya/same-class-number.

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I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.

Contents

The problem

Let F'/F be a finite extension of function fields associated to a cover $C' \to C$ of curves over finite fields. Let g, g' be the genera of F and F'. Let q, q' be the cardinalities of the base fields[1] of F, F'.

Let h, h' be the class numbers[2] of F and F'. The ratio h'/h equals $\#A(\mathbb{F}_q)$ for A the **Prym (abelian) variety** of C'/C, and hence an integer. Following Leitzel–Madan (1976), we ask: in what cases does $h'/h = 1$?

To make this a potentially finite problem, we only specify the isomorphism classes of F and F', not the inclusion (this only makes a difference when $g \leq 1$). We also ignore the trivial cases:

- $F' \cong F$;
- $g = g' = 0$.

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- **Solved** when F'/F is **constant** (i.e., $F' = F \cdot \mathbb{F}_{q'}$). We thus need only treat the case where F'/F is **geometric** (i.e., $q' = q$).
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- **Reduced to a finite computation**: the zeta functions $\zeta_F, \zeta_{F'}$ of F, F' form one of 208 known pairs. In all cases, $g \leq 7, g' \leq 13$.
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In this talk, I focus on the following statement.

Theorem

Let F'/F be a finite geometric extension of function fields with $q = 2, g > 1, h'/h = 1$. Then F'/F is cyclic.

A useful slogan here is

the most radical [extreme] **covers are radical** [cyclic]:

the class number condition puts severe pressure on point counts and splitting behavior in the extension, and cyclic covers are better able to withstand this pressure.

The proof will make free use of the zeta function constraint from the ANTS-XV work, cited hereafter as ANTS.

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This will leave unsettled the cases where $g = 6, 7$ and F'/F is unramified of degree 2, as in these cases we do not have complete tables of genus-g curves over $\mathbb{F}_2$ with which to perform a table lookup by zeta function.

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In this range, **in principle** it is possible to:

- find (by table lookup) all F with a particular ζ_F;[6]
- find all F'/F of degree d, e.g., using Bhargava's orbit parametrizations for $d \leq 5$;
- compute $\zeta_{F'}$ and confirm that the only extensions consistent with ANTS are cyclic.

However, it is not straightforward at present to make this practical. In particular orbit parametrizations have not been implemented for $d = 4, 5$, and do not exist at all for $d > 5$.

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A paradigm for $d \leq 6$

For $3 \leq d \leq 6$, we instead execute the following strategy.

- Given $\zeta_F, \zeta_{F'}$, identify the combinatorial options[7] for the splitting types of the low-degree places of F. E.g., there is **always** a degree-1 place of F that is inert in F'.
- Let F''/F be the Galois closure of F'/F. For each noncyclic candidate for $G = \mathrm{Gal}(F''/F) \subseteq S_d$, use the character theory of G to find other subfields of F''/F **and** compute how the various places of F split in these subfields. One important example is the **quadratic resolvent**, i.e., the fixed field of $G \cap A_d$.
- Identify isogeny factors of Jacobians for which we read off some Frobenius traces, then exhaust over Weil polynomials to obtain a contradiction.

This can be done uniformly using some simple SAGE code, but individual cases can also be analyzed by hand.

[7] These options differ slightly when the cover is ramified, but the strategy persists.

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The case $d = 7$

By ANTS, there is **one** possible pair $(\zeta_F, \zeta_{F'})$ with $(d, g, g') = (7, 2, 8)$. This does occur for a cyclic cover.

In principle one could execute the previous paradigm for $d = 7$. However, this requires working with the character tables of S_7 and its transitive subgroups (like $\mathrm{PSL}(2, 7)$), which seems infeasible.

Luckily, a method of Howe shows (in **very** special cases) that a curve whose zeta function “suggests an automorphism” actually has one.

This method also covers one missing step from ANTS: every curve $C/\mathbb{F}_2$ of genus 6 with $(\#C(\mathbb{F}_{2^i}))_{i=1}^6 = (0, 0, 0, 20, 15, 90)$ is a cyclic étale degree-5 cover of $y^2 + y = x^5$.

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By ANTS, there is **one** possible pair $(\zeta_F, \zeta_{F'})$ with $(d, g, g') = (7, 2, 8)$. This does occur for a cyclic cover.

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The case $d = 3$ (part 1)

For $d = 3$, we have to rule out $G = S_3$. Since F has a degree-1 place which is inert in F', the quadratic resolvent C''/A_3 is a geometric cover of C; let B be the Prym variety and $T_{B,q}$ its q-Frobenius trace.

Useful fact: if $\#C'(\mathbb{F}_q) = 0$, then $\#C''(\mathbb{F}_q) = 0$ and[8]

$$T_{A,q} = T_{J(C'),q} - T_{J(C),q} = \#C(\mathbb{F}_q)$$
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If F'/F is unramified, then $\dim(B) = g - 1$. By ANTS, $(g, g') \in \{(2,4),(3,7),(4,10)\}$.

- If $g = 2$, by ANTS, $T_{B,2} = -\#C(\mathbb{F}_2) = -3$, $\Rightarrow\Leftarrow$.
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For $d = 4$, we must rule out $G = D_4, S_4$. If $G = D_4$, we have a tower of geometric quadratic extensions $F'/E/F$ of relative class number 1, which we exclude using ANTS except for one case that arises as a cyclic cover.[10]

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Using ANTS to compute possible splitting sequences and then $T_{B_1,2^i}, T_{B_2,2^i}$ for $i = 1, \ldots, 7$ (!!)[12], we get $\Rightarrow\Leftarrow$.

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This leaves the cases $G = S_6$ and $G = \mathrm{PGL}(2, 5) \cong S_5$. We also split into cases based on whether the quadratic resolvent is constant or geometric.

In the case $G = S_6$, we look at the Prym B for $C''/\mathrm{PGL}(2, 5)$ over C; then $\dim(B) = 5$.

In the case $G = \mathrm{PGL}(2, 5)$, $C''/\mathrm{PGL}(2, 5)$ splits as a copy of C plus another cover; we look at the Prym B of the latter, so $\dim(B) = 4$.

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In all cases, we use to compute splitting sequences and then $T_{B,2^i}$ for $i \leq 6$, then $\Rightarrow\Leftarrow$.

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For $d = 6$, by ANTS, $g = 2$. We again use ANTS to rule out the possibility that F'/F has an intermediate subfield.

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