# Hypergeometric $L$-functions in average polynomial time

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slides at https://kskedlaya.org/slides/; see also arXiv:2005.13640, prerecorded talk

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The MIT campus sits on the traditional unceded territory of the Wampanoag Nation; we acknowledge the painful history of genocide and forced removal from this territory. The UCSD campus sits on the ancestral homelands of the Kumeyaay Nation; the Kumeyaay people continue to have an important and thriving presence in the region.

# Contents

# Computing an arithmetic $L$-function

An arithmetic $L$-function over $\mathbb{Q}$ of some degree $r$ generally has the form

$$\prod_p \det(1 - p^{-s} F_p)^{-1}$$

where for all but finitely many $p$, $F_p$ is some $r \times r$ matrix. Rewrite

$$\det(1 - p^{-s} F_p)^{-1} = \exp\left(\sum_{f=1}^{\infty} \frac{1}{f} p^{-fs} \operatorname{Trace}(F_p^f)\right);$$

to compute the Dirichlet series up to $X$, we need $\operatorname{Trace}(F_p^f)$ for all prime powers $p^f \leq X$.

We are interesting in computing the hypergeometric $L$-function associated to a hypergeometric datum $(\alpha, \beta) \in (\mathbb{Q} \cap [0,1))^{r \times 2}$, for which $\operatorname{Trace}(F_p^f)$ is computed by a finite hypergeometric sum. In this paper, we focus on $f = 1$ and compute this trace modulo $p$.

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# Finite hypergeometric sums

Using Gross–Koblitz to compute Gauss sums in the Beukers–Cohen–Mellit formula using the Morita $p$-adic Gamma function $\Gamma_p$, we get for $q = p$

$$\mathrm{Trace}(F_p) = H_p\left(\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\middle| z\right) := \frac{1}{1-p}\sum_{m=0}^{p-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}p^{D+\xi_m(\beta)} \left(\prod_{j=1}^{r}\frac{\Gamma_p(\alpha_j+\frac{m}{1-p})/\Gamma_p(\alpha_j)}{\Gamma_p(\beta_j+\frac{m}{1-p})/\Gamma_p(\beta_j)}\right)[z]^m$$

where $\eta_m, \xi_m, D$ are some combinatorial invariants of $\alpha, \beta$ and $[z] \in \mathbb{Z}_p^\times$ is the unique $(p-1)$-st root of unity congruent to $z$ modulo $p$. (We rig up $D$ to ensure $\eta_m(\alpha) - \eta_m(\beta) + D + \xi_m(\beta) \geq 0$; since $\Gamma_p$ takes values in $\mathbb{Z}_p^\times$, everything in sight is in $\mathbb{Z}_p$ rather than $\mathbb{Q}_p$.)

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# Quadratic versus linear complexity

The implementations in `Magma` and `Sage` compute $H_p\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\middle| z\right)$ one $p$ at a time. Since the sum is over $O(p)$ terms, computing all prime Dirichlet coefficients up to $X$ requires $O(\frac{X^2}{\log X})$ arithmetic operations.

In our paper, we use the method of remainder forests (cf. Sutherland's paper) to amortize the computation over all $p \leq X$. This reduces the complexity to $O(X \log^3 X)$ (for fixed $\alpha, \beta$).

Reminder: we are only computing $H_p\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\middle| z\right) \pmod{p}$. However, we expect that one can work modulo $p^e$ with similar complexity (times some power of $e$). It would still remain to compute $H_{p^f}\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\middle| z\right)$ for all $p^f \leq X$ with $f \geq 2$; this requires $O(\frac{X^{3/2}}{\log X})$ as written, but other techniques can reduce this to $O(X \log^? X)$ even without amortization.

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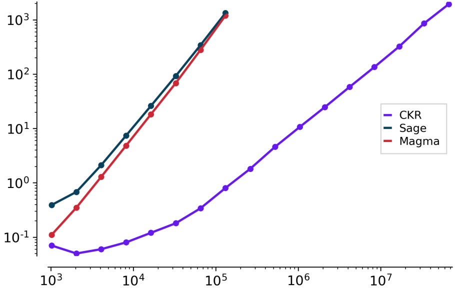
# Timings

In this example $\alpha = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}), \beta = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}), z = \frac{1}{5}$. This $L$-function has weight 1, so $H_p\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle| z\right)$ is uniquely determined by its reduction mod $p$. (See §5.4 of the paper for more implementation details, and §5.5 for a worked example.)

| $X$ | Amortized | Sage | Magma |
|---|---|---|---|
| $2^{10}$ | 0.07s | 0.39s | 0.11s |
| $2^{11}$ | 0.05s | 0.68s | 0.35s |
| $2^{12}$ | 0.06s | 2.12s | 1.29s |
| $2^{13}$ | 0.08s | 7.39s | 4.83s |
| $2^{14}$ | 0.12s | 26.0s | 18.2s |
| $2^{15}$ | 0.18s | 92.3s | 68.4s |
| $2^{16}$ | 0.34s | 343s | 280s |
| $2^{17}$ | 0.80s | 1328s | 1190s |




| $X$ | $2^{18}$ | $2^{19}$ | $2^{20}$ | $2^{21}$ | $2^{22}$ | $2^{23}$ | $2^{24}$ | $2^{25}$ | $2^{26}$ |
|---|---|---|---|---|---|---|---|---|---|
| Amortized | 1.81s | 4.59s | 10.7s | 24.6s | 58.0s | 135s | 322s | 857s | 1948s |

# Contents

# Setup

Modulo $p$, the trace formula becomes

$$H_p\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| z\right) \equiv \sum_{m=0}^{p-2} \pm p^* \left(\prod_{j=1}^{r} \frac{\Gamma_p(\alpha_j+m)/\Gamma_p(\alpha_j)}{\Gamma_p(\beta_j+m)/\Gamma_p(\beta_j)}\right) z^m \pmod p.$$

Call the $m$-th summand $P_m$. Suppose we had $f(m), g(m) \in \mathbb{Z}[m]$ so that

$$P_{m+1} \equiv \frac{f(m)}{g(m)} P_m \pmod p.$$

We could then set

$$B(m) := \begin{pmatrix} g(m) & 0 \\ g(m) & f(m) \end{pmatrix} = g(m) \begin{pmatrix} 1 & 0 \\ 1 & f(m)/g(m) \end{pmatrix}$$

and then use remainder products to compute

$$B(0)\dots B(p-2) \equiv g(0)\cdots g(p-2) \begin{pmatrix} 1 & 0 \\ \sum_{m=0}^{p-2} P_m & P_{p-1} \end{pmatrix} \pmod p.$$

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# Two related issues

- The factor $\pm p^*$ is determined by the **zigzag function*** at $\frac{m}{p-1}$:

$$Z_{\alpha,\beta} : [0,1] \to \mathbb{Z}, \quad Z_{\alpha,\beta}(x) := \#\{j : \alpha_j \le x\} - \#\{j : \beta_j \le x\}.$$

This creates a "discontinuity" when $\frac{m}{p-1}$ passes through $\alpha_j$ or $\beta_j$.

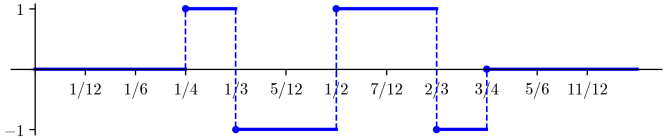


Figure: $Z_{\alpha,\beta}(x)$ for $\alpha = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}), \beta = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

- Similar "discontinuities" arise from the functional equation for $\Gamma_p$:

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & x \notin p\mathbb{Z}_p \\ -\Gamma_p(x) & x \in p\mathbb{Z}_p \end{cases}$$

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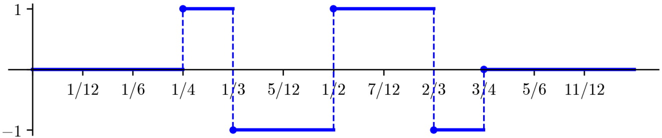


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* $Z_{\alpha,\beta}$ also determines the weight and Hodge numbers of the $L$-function.

# Resolution of the issues

We resolve both issues by "ferrying".†

- We break the summation at $\lfloor \alpha_j(p-1) \rfloor, \lfloor \beta_j(p-1) \rfloor$, and separate primes into classes modulo $\mathrm{lcd}(\alpha, \beta)$.
- Within each range and congruence class, we do a single amortized computation of matrix products.
- We then do non-amortized computations of transition matrices to "portage" or "ferry" across the breaks.

For each $p$, we put the ranges and transitions together to obtain a product computing a scalar multiple of $\begin{pmatrix} 1 & 0 \\ \sum_{m=0}^{p-2} P_m & P_{p-1} \end{pmatrix} \pmod{p}$.


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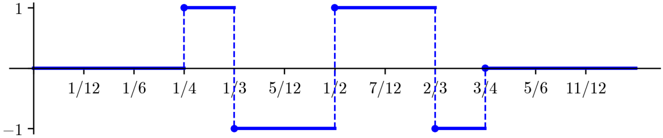


In particular, computing $H_p$ modulo $p$ is enough to determine it exactly. Denote the intervals we see by $I_0, \ldots, I_5$.

Since we are only working modulo $p$, the only intervals that contribute to the sum are $I_2 = (\frac{1}{3}, \frac{1}{2})$ and $I_4 = (\frac{2}{3}, \frac{3}{4})$. **However**, we do still have to compute over the other integrals in order to update the product!

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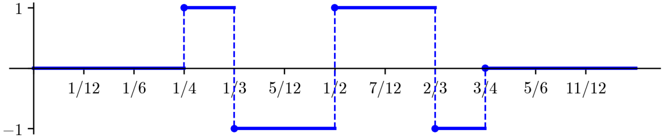


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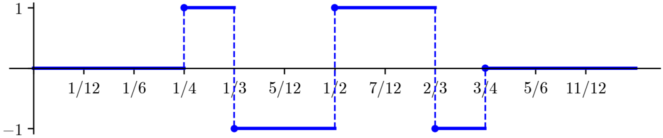


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# Amortized products

For simplicity, we focus on the case $p \equiv 7 \pmod{12}$. In the intervals that contribute to the sum, we take in the matrix product

$$\begin{aligned} f_{2,7}(k) &= 5184k^4 + 8640k^3 + 4428k^2 + 852k + 55, \\ g_{2,7}(k) &= 25920k^4 + 69120k^3 + 63360k^2 + 23040k + 2880, \\ f_{4,7}(k) &= 5184k^4 + 12096k^3 + 9612k^2 + 2820k + 175, \\ g_{4,7}(k) &= 25920k^4 + 86400k^3 + 106560k^2 + 57600k + 11520. \end{aligned}$$

Suppose we did the remainder forest and then took $p = 67$. We'd see

$$S_2(67) = \begin{pmatrix} 65 & 0 \\ 34 & 5 \end{pmatrix}, \qquad S_4(67) = \begin{pmatrix} 54 & 0 \\ 25 & 41 \end{pmatrix}.$$

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# More amortized products and the portage

In order to compute the correct sum, we also do similar computations over the other intervals. At $p = 67$, we get

$$S_0(67) = \begin{pmatrix} 38 & 0 \\ 0 & 62 \end{pmatrix}, \qquad S_1(67) = \begin{pmatrix} 50 & 0 \\ 0 & 47 \end{pmatrix},$$

$$S_3(67) = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}, \qquad S_5(67) = \begin{pmatrix} 1 & 0 \\ 0 & 38 \end{pmatrix}.$$

For the "ferries", we work directly with $p = 67$ to compute

$$T_0(67) = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}, \quad T_1(67) = \begin{pmatrix} 1 & 0 \\ 0 & 31 \end{pmatrix}, \quad T_2(67) = \begin{pmatrix} 1 & 0 \\ -1 & 12 \end{pmatrix},$$

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# More amortized products and the portage

In order to compute the correct sum, we also do similar computations over the other intervals. At $p = 67$, we get

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# A worked example (part 4)

Putting the product together, we get

$$S(67) = T_0(67)S_0(67)\cdots T_5(67)S_5(67) = \begin{pmatrix} 21 & 0 \\ 33 & 21 \end{pmatrix}$$

so $H_{67}\left(\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\middle|\tfrac{1}{5}\right) \equiv \frac{33}{21} \equiv 59 \pmod{67}$. This checks with `Magma` and `Sage`:

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H := HypergeometricData([[1/4,1/2,1/2,3/4],[1/3,1/3,2/3,2/3]]);
HypergeometricTrace(H, 5, 67);
-8

sage: from sage.modular.hypergeometric_motive \
....: import HypergeometricData as Hyp
sage: H = Hyp(alpha_beta=([1/4,1/2,1/2,3/4],[1/3,1/3,2/3,2/3]))
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# Contents

# Raising the modulus

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- The general formula has $[z]$ (the $(p-1)$-st root of unity congruent to $z$ modulo $p$) instead of $z$. One can compute $[z]$ modulo $p^e$ (e.g., by a Newton-Raphson iteration) but this does not integrate well into the amortization.
- The general formula has $\Gamma_p(\alpha_j + \frac{m}{1-p})$ rather than $\Gamma_p(\alpha_j + m)$. One can compute $\Gamma_p$ using its Mahler expansion in a residue disc, but it takes $O(p)$ complexity to compute the coefficients (e.g., modulo $p^2$ one needs $(p-1)! \pmod{p^2}$ as in a search for Wilson primes).

To deal with the first issue, one can use Harvey's "generic prime" strategy: replace $\mathbb{Z}[m]$ with $\mathbb{Z}[m,x]/(x^e)$ where $x$ is a proxy for $[z]-z$.

To deal with the second issue, we replace $p$ by a second nilpotent variable $y$, and integrate Mahler coefficients into the amortized computation.

We have not tried this! But it should work well in practice for small $e$.

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We also need a plan for dealing with the $p^f$-Frobenius traces for $f > 1$.

For Dirichlet coefficients up to $X$, there are $O(\frac{X^{1/2}}{\log X})$ of these, and the primes involved are $O(X^{1/2})$. So we don't need to amortize if we can reduce the individual complexity from $O(p^f)$ to $O(p)$.

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