Banach bundles

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*The UCSD campus sits on the ancestral homelands of the Kumeyaay Nation; the Kumeyaay people continue to have an important and thriving presence in the region.

Sheaves in algebraic geometry

For any ring A, the category of A-modules is equivalent to the category of **quasicoherent sheaves** on the topological space $\operatorname{Spec} A$.

A special case of this is that the category of **finite projective** A-modules is equivalent to the category of **locally finite free quasicoherent sheaves** on $\operatorname{Spec} A$. These also correspond to **vector bundles** over $\operatorname{Spec} A$, and I will frequently confuse the two.

Classical p-adic geometry

Classical p-adic analytic geometry is analogous to the geometry of schemes **locally of finite type** over a field k. These schemes are locally of the form $\operatorname{Spec} A$ where A is a quotient of a polynomial ring $k[x_1, \dots, x_n]$.

Classical p-adic analytic geometry involves spaces locally associated to a quotient of a ring of the form $K\langle x_1, \dots, x_n \rangle$ where K is a field complete with respect to a nonarchimedean absolute value (e.g., $\mathbb{Q}_p$). Here $K\langle x_1, \dots, x_n \rangle$ denotes the subring of $K[[x_1, \dots, x_n]]$ consisting of series which converge for $|x_1|, \dots, |x_n| \leq 1$; it is the completion of $K[x_1, \dots, x_n]$ for the Gauss norm.

Models of nonarchimedean geometry

There are a variety of recipes to turn such rings into spaces (which all give essentially the same theory). For example, say $A = \mathbb{Q}_p\langle x_1, \dots, x_n \rangle$.

- Tate: maximal ideals of A. This is a subset of $\operatorname{Spec} A$, but with a different Grothendieck topology.
- Raynaud: the formal scheme $\operatorname{Spf} \mathbb{Z}_p[x_1, \dots, x_n]^\wedge_{(p)}$ with its topology **modulo** inverting blowups in the special fiber.
- Berkovich: multiplicative seminorms on A. This set (the Gelfand spectrum) has a "reasonable" ordinary topology, but in general we need a finer Grothendieck topology.
- Huber: continuous valuations on A (not necessary real-valued). This set carries a natural **ordinary** topology; it is even a **spectral space**.

Of these, only Huber's approach extends to more general topological rings (Huber's f-adic rings). However, one can do "Huber + Raynaud" (Fujiwara–Kato) or "Huber + Berkovich" (KSK).

Sheaves in classical p-adic geometry

Theorem (Tate, Kiehl)

For A an affinoid algebra over K corresponding to the analytic space X, the category of finite A-modules is equivalent to the category of coherent sheaves on X. Moreover, for any such sheaf $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Can this be extended to more general (topological) A-modules and sheaves on X?

What about more general analytic spaces, which need not be topologically finite type over a field?

The analytic FF curve

Let F be a perfect field of characteristic $p > 0$ complete with respect to a nontrivial absolute value (i.e., a **perfectoid field** of characteristic p).

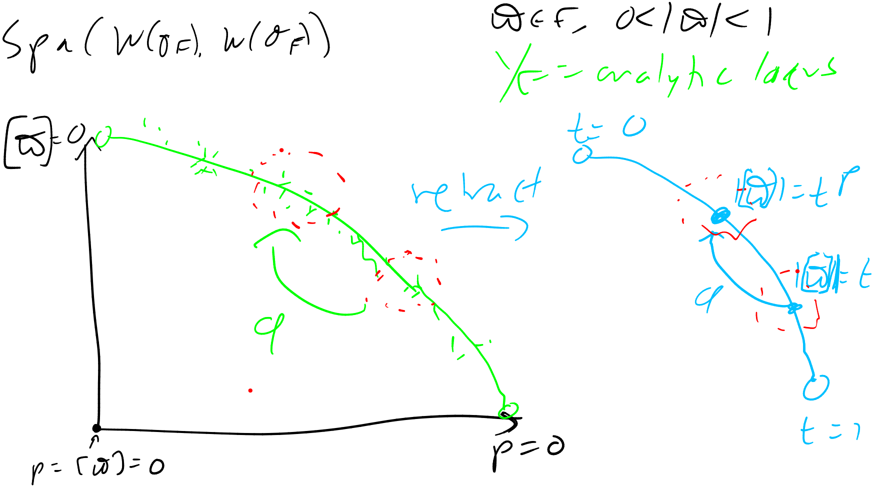
The space Y_F is the "analytification" of $W(\mathfrak{o}_F)$. In Huber's language, it is the subspace of $\mathrm{Spa}(W(\mathfrak{o}_F), W(\mathfrak{o}_F))$ consisting of valuations v with $0 < v(x) < 1$ for all x in the maximal ideal of $W(\mathfrak{o}_F)$.

The Frobenius map $\varphi : W(\mathfrak{o}_F) \to W(\mathfrak{o}_F)$ induces a properly discontinuous[†] automorphism of Y_F. The **analytic Fargues–Fontaine curve** is the quotient $X_F^{\mathrm{an}} = Y_F/\varphi$.

This space is (quasi)compact and noetherian (!).

[†]Reminder: this means that any point has a neighborhood whose translates by distinct powers of φ are pairwise disjoint.

Illustration



The schematic FF curve

We may write down vector bundles on X_F^{an} by writing down vector bundles on Y_F equipped with an action of φ. In particular, taking the trivial bundle with the action of φ on a free generator being multiplication by p^{-n} gives a line bundle $\mathcal{O}(n)$.

The **schematic FF curve** is the scheme

$$X_F = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma(X_F^{\mathrm{an}}, \mathcal{O}(n))\right).$$

This scheme is noetherian (!) and regular of dimension 1 (!!). However, it is **not** of finite type over $\mathbb{Q}_p$, or indeed over any field at all!

By the universal property of affine schemes, there is a natural morphism $X_F^{\mathrm{an}} \to X_F$ in the category of locally ringed spaces.

GAGA for vector bundles (and coherent sheaves)

Theorem (GAGA, Serre, Grothendieck)

Let X be a proper scheme of finite type over $\mathbb{C}$. Then the morphism $X^{\mathrm{an}} \to X$ of locally ringed spaces induces an equivalence of categories for coherent sheaves. Moreover, sheaf cohomology groups are preserved.

Theorem (Nonarchimedean GAGA)

Let X be a proper scheme of finite type over a nonarchimedean field K. Then the morphism $X^{\mathrm{an}} \to X$ of locally ringed spaces induces an equivalence of categories for coherent sheaves. Moreover, sheaf cohomology groups are preserved.

Theorem (KSK–Liu)

Pullback along the morphism $X_F^{\mathrm{an}} \to X_F$ of locally ringed spaces induces an equivalence of categories for coherent sheaves. Moreover, sheaf cohomology groups are preserved.

The key lemma for GAGA

Lemma (KSK–Ruochuan Liu)

For any vector bundle (or coherent sheaf) $\mathcal{F}$ on X_F^{an}, for all $n \gg 0$,

(a) $H^1(X_F^{\mathrm{an}}, \mathcal{F}(n)) = 0$;

(b) $\mathcal{F}(n)$ *is generated by global sections.*

In other words, $\mathcal{O}(1)$ is an **ample** line bundle on X_F^{an}.

The proof of (a) amounts to a careful use of the Banach contraction mapping theorem. From (a) it is relatively straightforward to deduce (b).

Warning: $H^0(X_F^{\mathrm{an}}, \mathcal{F}(n))$ is not a finite-dimensional $\mathbb{Q}_p$-vector space! It does have a weaker finiteness property, of being a **Banach–Colmez space**.

Why this matters

There is a classification[‡] of vector bundles on X_F akin to that for $\mathbb{P}^1$ (K, Hartl–Pink, Fargues–Fontaine, Fargues–Scholze). The semistable bundles of degree 0 are related to p-adic Galois representations (as in Narasimhan–Seshadri over $\mathbb{C}$).

Using the GAGA construction, one can construct **moduli stacks of vector bundles** on X_F (analogous to the use of GIT quotients to construct moduli spaces of vector bundles on curves). These can then be used to construct **moduli spaces of local shtukas** for use in the local Langlands correspondence (Fargues–Scholze).

[‡] For F algebraically closed, not just perfectoid.

The goal of this talk

In classical p-adic (or complex geometry), **analytification** is a functor from schemes locally of finite type over a nonarchimedean field K to analytic spaces over K. If X is such a scheme, then for any analytic space Y, any morphism $Y \to X$ of locally ringed spaces factors through X^{an}.

This functor does **not** apply to X_F because the latter is not of finite type over any field. However, we will show that the GAGA statement for X_F is not isolated! Putting it in the right context is a question for future work.

Relative versions

GAGA for p-adic analytic spaces can also be formulated for a scheme over an affinoid algebra (Kopf, Conrad).

GAGA for FF curves can also be formulated for the **relative FF curve** over a **perfectoid space** (KSK–Liu). However, the latter is **not** locally noetherian, so we only do the vector bundle case. (This is relevant to studying relative p-adic Galois representations, a/k/a $\mathbb{Q}_p$-**local systems**.)

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Banach modules over a Huber ring

Let A be a complete Tate Huber ring. One way to get one of these is to take a (commutative) **Banach ring**, i.e., a ring complete for a submultiplicative norm **and** containing a topologically nilpotent unit, and retain the underlying topology. (E.g., an affinoid algebra.)

A **Banach module** is an A-module which admits a topology with respect to which it is complete and metrizable. (By the Banach open mapping theorem, this topology is unique if it exists.)

Let $\underline{\mathrm{BMod}}_A$ be the category of Banach modules with continuous homomorphisms.

Projective objects in the category of Banach modules

As usual, a **projective** object P of $\underline{\mathrm{BMod}}_A$ is characterized by this diagram:

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ \uparrow & \nearrow & & & \\ P & & & & \end{array}$$

For example, every object of $\underline{\mathrm{BMod}}_A$ is a quotient of a **topologically free** A-module (a completion of a free module). Hence:

- The category $\underline{\mathrm{BMod}}_A$ has enough projectives.
- An object of $\underline{\mathrm{BMod}}_A$ is projective iff it is a summand of a topologically free module.

Let $\underline{\mathrm{BPMod}}_A$ be the full subcategory of projective objects of $\underline{\mathrm{BMod}}_A$.

Sheafy Huber rings

One difficulty of Huber's theory of adic spaces is that, for A a complete Tate Huber ring (and A^+ a ring of integral elements), the adic spectrum $\mathrm{Spa}(A, A^+)$ carries a natural **structure presheaf** which is **not** necessarily a sheaf.

Concretely, this means that for $f, g \in A$ generating the unit ideal, the Čech sequence

$$0 \to A \to \frac{A\langle T\rangle}{\overline{(f - Tg)}} \oplus \frac{A\langle U\rangle}{\overline{(g - Uf)}} \to \frac{A\langle T, U\rangle}{\overline{(f - Tg, g - Uf, TU - 1)}} \to 0$$

is not necessarily exact. (Note the closures of ideals.)

We say that A is **sheafy** if the structure presheaf $\mathcal{O}$ on $\mathrm{Spa}(A, A^+)$ is a sheaf. In this case, $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}) = 0$ for all $i > 0$, so the above sequence is exact; moreover, the closures are not required (KSK–Liu).

Banach bundles and Banach sheaves

Let A be a sheafy complete Tate Huber ring and put $X = \mathrm{Spa}(A, A^+)$ (for some choice of A^+). For every $M \in \underline{\mathrm{BMod}}_A$, we can define a presheaf $\tilde{M} = M \widehat{\otimes}_A \mathcal{O}$ on X. A **Banach bundle** is a sheaf on X which is locally isomorphic to the presheaf associated to some **projective** Banach module in this way.

Theorem (KSK; after Tate, Kiehl, KSK–Liu)

The category $\underline{\mathrm{BPMod}}_A$ is equivalent to the category of Banach bundles on X (via the global sections functor). Moreover, for any such sheaf $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Aside for experts: Banach bundles are also acyclic sheaves for the étale topology, the pro-étale topology, and the v-topology.

Warning: here be dragons!

In general, it is quite difficult to decide whether a given Huber ring is sheafy. Examples include affinoid algebras (Tate) and **perfectoid rings** (KSK–Liu, Scholze). It is a question of active interest to develop useful criteria for sheafiness (e.g., Hansen–KSK).

Even when A is sheafy, without some noetherian hypothesis, a localization map $A \to B$ can fail to be flat. However, they are in some sense "topologically flat" (KSK–Liu).

One way to circumvent these issues is to work in a suitably "derived" framework (Bambozzi–Kremnitzer, Clausen–Scholze). Here we limit ourselves to situations where we can avoid this; our results should embed into such a treatment as a special case where everything is "acyclic".

Reminder: adic and schematic FF curves

Let F be a perfect field of characteristic p complete for a nontrivial absolute value.[¶] As in part 1, define the **adic FF curve** X_F^{an} and the **schematic FF curve** X_F. Reminder: Y_F is the subspace of $\mathrm{Spa}(W(\mathfrak{o}_F), W(\mathfrak{o}_F))$ obtained by omitting the fixed points of φ; X_F^{an} is the quotient of Y_F by the (properly discontinuous) action of φ; and

$$X_F = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma(X_F^{\mathrm{an}}, \mathcal{O}(n))\right)$$

where $\mathcal{O}(n)$ is the line bundle corresponding to the trivial line bundle on Y_F with the action of φ on a free generator being multiplication by p^{-n}.

[¶] That is, F is a **perfectoid field** of characteristic p.

Products of FF curves

Let d be a positive integer (the case $d = 1$ will reproduce the previous slide). Let $(X_F^d)^{\mathrm{an}}$ be the d-fold product of X_F^{an} over $\mathbb{Q}_p$. This is a **sousperfectoid** space (Hansen–KSK), hence a genuine adic space. It is **not** locally noetherian, so we do not try to study coherent sheaves on it.

Define also the scheme

$$X_F^d = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma((X_F^d)^{\mathrm{an}}, \mathcal{O}(n) \boxtimes \cdots \boxtimes \mathcal{O}(n))\right)$$

where $\boxtimes$ denotes the external product.

The main theorem

Theorem (KSK)

Pullback along the morphism $(X_F^d)^{\mathrm{an}} \to X_F^d$ of locally ringed spaces induces an equivalence of categories for vector bundles. Moreover, sheaf cohomology groups are preserved.

Open question: can one define a meaningful notion of "finite type" (in quotes!!) that would include closed analytic subspaces of $(X_F^d)^{\mathrm{an}}$? (Remember, X_F^{an} is not itself topologically of finite type over any field.)

The proof of this will require use of Banach bundles!

Relative Fargues–Fontaine curves

Let S be a perfectoid space of characteristic p (e.g., $\mathrm{Spa}(F, F^\circ)$).

Let Y_S be the subspace of $\mathrm{Spa}(W(S^\circ), W(S^\circ))$ obtained by removing the fixed points of the Frobenius map $\varphi : W(S^\circ) \to W(S^\circ)$. Then φ induces a properly discontinuous automorphism of Y_S; the **analytic relative Fargues–Fontaine curve** is $X_S^{\mathrm{an}} = Y_S/\varphi$.

The **schematic relative Fargues–Fontaine curve** is the scheme

$$X_S = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma(X_S^{\mathrm{an}}, \mathcal{O}(n))\right)$$

where again $\mathcal{O}(n)$ is the line bundle corresponding to the trivial line bundle on Y_S with the action of φ on a free generator being multiplication by p^{-n}.

Warning: This construction does "lie over S" but only in a quite subtle way (e.g., in Scholze's theory of **diamonds**).

Comparison with the $d = 1$ case

We are trying to show...

Theorem

Pullback along the morphism $(X_F^d)^{\mathrm{an}} \to X_F^d$ of locally ringed spaces induces an equivalence of categories for vector bundles. Moreover, sheaf cohomology groups are preserved.

It would be enough to show...

Lemma

For any vector bundle $\mathcal{F}$ on $(X_F^d)^{\mathrm{an}}$, for all $n \gg 0$,

$$\mathcal{F}(n, \dots, n) = \mathcal{F} \otimes (\mathcal{O}(n) \boxtimes \cdots \boxtimes \mathcal{O}(n))$$

is generated by global sections.

Comparison with the $d = 1$ case

We are trying to show...

Lemma

For any vector bundle $\mathcal{F}$ on $(X_F^d)^{\mathrm{an}}$, for all $n \gg 0$, $\mathcal{F}(n, \dots, n)$ is generated by (finitely many) global sections.

But a direct attempt to emulate the $d = 1$ case only gives us...

Lemma

For any vector bundle $\mathcal{F}$ on $(X_F^d)^{\mathrm{an}}$, for all $n \gg 0$, $H^d((X_F^d)^{\mathrm{an}}, \mathcal{F}(n, \dots, n)) = 0$.

... and this is not enough: we need to get from H^d down to H^1.

Enter Banach bundles

We are trying to show...

Lemma

For any vector bundle $\mathcal{F}$ on $(X_F^d)^{\mathrm{an}}$, for all $n \gg 0$, $\mathcal{F}(n, \dots, n)$ is generated by (finitely many) global sections.

It will be enough to show...

Lemma

For any[a] Banach bundle $\mathcal{F}$ on $(X_F^d)^{\mathrm{an}}$, for all $n \gg 0$, $\mathcal{F}(n, \dots, n)$ is (topologically) generated by global sections.

[a] There is an uniformity condition needed here, which is automatic for vector bundles, to rule out something like a completion of $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \cdots$. I'll gloss over this hereafter.

Enter the relative setting

We are trying to show...

Lemma

For any Banach bundle $\mathcal{F}$ on $(X_F^d)^{\mathrm{an}}$, for all $n \gg 0$, $\mathcal{F}(n, \dots, n)$ is (topologically) generated by global sections.*

It will be enough to show...

Lemma

Supose that S is affinoid perfectoid. For any Banach bundle $\mathcal{F}$ on $(X_S^d)^{\mathrm{an}}$, for all $n \gg 0$, $\mathcal{F}(n, \dots, n)$ is (topologically) generated by global sections.*

Setup for induction on d (dévissage)

We are trying to show...

Lemma

Supose that S is affinoid perfectoid. For any Banach bundle $\mathcal{F}$ on $(X_S^d)^{\mathrm{an}}$, for all $n \gg 0$, $\mathcal{F}(n, \dots, n)$ is topologically generated by global sections.*

We will use...

Lemma (KSK)

Let F be a completed algebraic closure of $\mathbb{Q}_p$. Then $(X_S^d)^{\mathrm{an}} \times_{\mathbb{Q}_p} F$ is isomorphic to the relative FF curve over $(X_S^{d-1} \times_{\mathbb{Q}_p} F)^\flat$ where $\flat$ denotes the tilt *of the perfectoid space $X_S^{d-1} \times_{\mathbb{Q}_p} F$.*

Induction on d (dévissage)

We are trying to show...

Lemma

Supose that S is affinoid perfectoid. For any Banach bundle $\mathcal{F}$ on $(X_S^d)^{\mathrm{an}}$, for all $n \gg 0$, $\mathcal{F}(n, \ldots, n)$ is topologically generated by global sections.*

We can reduce this to the corresponding statement with $(X_S^d)^{\mathrm{an}}$ replaced by $(X_S^d)^{\mathrm{an}} \times_{\mathbb{Q}_p} F$. We can then show something in the $d = 1$ case...

Lemma

Supose that S is affinoid perfectoid. For any Banach bundle $\mathcal{F}$ on $X_S^{\mathrm{an}} \times_{\mathbb{Q}_p} F$, for all $n \gg 0$, $H^1(X_S^{\mathrm{an}} \times_{\mathbb{Q}_p} F, \mathcal{F}(n)) = 0$.*

Induction on d (dévissage)

From the previous lemma...

Lemma

Supose that S is affinoid perfectoid. For any Banach bundle $\mathcal{F}$ on $X_S^{\mathrm{an}} \times_{\mathbb{Q}_p} F$, for all $n \gg 0$, $H^1(X_S^{\mathrm{an}} \times_{\mathbb{Q}_p} F, \mathcal{F}(n)) = 0$.*

... we can deduce...

Lemma

For any Banach bundle $\mathcal{F}$ on $X_S^{\mathrm{an}} \times_{\mathbb{Q}_p} F$, for all $n \gg 0$, $H^0(X_S^{\mathrm{an}} \times_{\mathbb{Q}_p} F, \mathcal{F}(n))$ is a Banach module whose formation commutes with arbitrary (completed) base change.*

Now given a Banach bundle $\mathcal{F}$ on $(X_S^d)^{\mathrm{an}}$, we'd like to "push forward" some twist to obtain a Banach bundle on $(X_S^{d-1})^{\mathrm{an}}$; **but** the map $X_S^{\mathrm{an}} \to S$ is only topological. However, it does admit a genuine section and we can **pull back** (no quotes) along (some thickening of) that section.

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