Several forms of Drinfeld's lemma

Kiran S. Kedlaya

Department of Mathematics, University of California, San Diego
kedlaya@ucsd.edu
http://kskedlaya.org/slides/

Recent Advances in Modern p-Adic Geometry
virtual seminar
November 12, 2020

Supported by NSF (grant DMS-1802161) and UC San Diego (Warschawski Professorship).

Contents

References for this section

Eike Lau, On generalised $\mathscr{D}$-shtukas, PhD thesis (Bonn, 2004), pdf.

KSK, Sheaves, stacks, and shtukas, Arizona Winter School 2017 (pdf).

Lecture 4

Setup: a formal quotient by Frobenius

X = a scheme over $\mathbb{F}_p$
k = an algebraically closed field of characteristic p
$X_k = X \times_{\mathbb{F}_p} k$ $\longrightarrow X \times_{\mathbb{F}_p} k$ "partial Frobenius"
φ_k = the pullback to X_k of the absolute Frobenius on $\operatorname{Spec} k$ $\neq \varphi_{X_k}$

We will consider "X_k/φ_k" is a formal quotient: an object of some type over X_k/φ_k is an object of the same type over X_k equipped with an isomorphism with its φ_k-pullback.

Coherent sheaves

Theorem (Drinfeld, Lau) projective

For $X/\mathbb{F}_p$ ~~of finite type~~, the base extension functor

$$\text{(coherent sheaves on } X) \to \text{(coherent sheaves on } X_k/\varphi_k)$$

H^i $\qquad$ $H^i_{\varphi_K}$

is an equivalence of categories and preserves cohomology. $\quad K^{\varphi_K} = \mathbb{F}_p$

Idea of proof: ~~reduce to X projective~~, trivialize φ_k-action on $H^0(X_K, \mathcal{E}(n))$.

finite-dim K-vector spaces
with isomorphism with φ_K-pullbacks
$\wr\wr$ Katz-Lang
Fin dim $\mathbb{F}_p$-vector spaces

Finite étale covers and profinite fundamental groups

Corollary

For any X, $\mathbf{FEt}(X) \to \mathbf{FEt}(X_k/\varphi_k)$ is an equivalence.

Corollary

For X connected, X_k/φ_k is connected and for any geometric point $\overline{x} \to X_k$, $\pi_1^{\mathrm{prof}}(X_k/\varphi_k, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X, \overline{x})$.

Warning: in general $\pi_0(X_k) \neq \pi_0(X)$. For example, if $X = \operatorname{Spec} \ell$ is a geometric point, $\pi_0(X_k) \cong \widehat{\mathbb{Z}}$ indexed by identifications of the copies of $\overline{\mathbb{F}}_p$ in k and ℓ; but φ_k acts on $\pi_0(X_k)$ by translation by $\mathbb{Z}$.

Finite étale covers and profinite fundamental groups

Corollary

For any X, $\mathbf{FEt}(X) \to \mathbf{FEt}(X_k/\varphi_k)$ is an equivalence.

Corollary

For X connected, X_k/φ_k is connected and for any geometric point $\overline{x} \to X_k$, $\pi_1^{\mathrm{prof}}(X_k/\varphi_k, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X, \overline{x})$.

Warning: in general $\pi_0(X_k) \neq \pi_0(X)$. For example, if $X = \operatorname{Spec} \ell$ is a geometric point, $\pi_0(X_k) \cong \widehat{\mathbb{Z}}$ indexed by identifications of the copies of $\overline{\mathbb{F}}_p$ in k and ℓ; but φ_k acts on $\pi_0(X_k)$ by translation by $\mathbb{Z}$.

Products of two (or more) fundamental groups

case where $X_2 = \operatorname{Spec} \overline{k}$ is previous version

Corollary

For X_1, X_2 two connected qcqs $\mathbb{F}_p$-schemes, put $X = X_1 \times_{\mathbb{F}_p} X_2$ and let $\varphi_1, \varphi_2 : X \to X$ be the partial Frobenius maps. Then X/φ_2 is connected qc ~~*qcqs,*~~ *and for any geometric point $\overline{x} \to X$,*

$$\pi_1^{\mathrm{prof}}(X/\varphi_2, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x}).$$

In contrast, if $X_1, X_2 \to \operatorname{Spec} \mathbb{C}$ finite type connected

$$\pi_1^{\mathrm{prof}}(X_1 \times_{\mathbb{C}} X_2, \overline{x}) \xrightarrow{\sim} \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_1^{\mathrm{prof}}(X_2, \overline{x})$$

$\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}}_p})$ is huge!

$X_1 \times_{\mathbb{F}_p} X_2 \times_{\mathbb{F}_p} X_3 / \langle \varphi_2, \varphi_3 \rangle$

Open subschemes and étale sheaves

Corollary relatively

For any X, quasicompact open subschemes of X and X_k/φ_k are the same.

Corollary of finite type

For X any $\mathbb{F}_p$-scheme and $\ell \neq p$ prime,

still works if $\ell = p$

$$(\text{lisse } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X) \to (\text{lisse } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X_k/\varphi_k)$$

$$(\text{constructible } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X) \to (\text{constructible } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X_k/\varphi_k)$$

are equivalences of categories and preserve cohomology. (And so on.)

Context: shtukas and excursion operators

These constructions are used to describe **excursion operators** on moduli stacks of shtukas, in order to describe the Langlands correspondence per V. Lafforgue. (See last week's seminar!)

Similarly, other forms of Drinfeld's lemma are needed to do likewise for local Langlands in mixed characteristic, or for p-adic coefficients in positive characteristic.

Contents

References for this section

Carter–KSK–Zábrádi, Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules, arXiv:1808.03964v2 (2020).

KSK, Sheaves, stacks, and shtukas, Arizona Winter School 2017 (pdf).

KSK, Simple connectivity of Fargues-Fontaine curves, arXiv:1806.11528v3 (2018).

Scholze–Weinstein, *Berkeley Lectures on p-adic Geometry* (pdf).

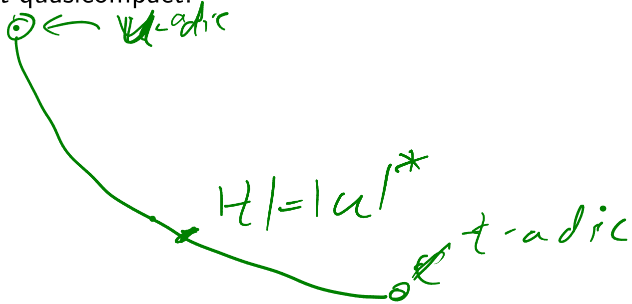
Absolute products of perfectoid spaces

Let $\mathbf{Pfd}$ be the category of perfectoid spaces in characteristic p. This category admits absolute products.

For example, if $X_1 = \operatorname{Spa} \mathbb{F}_p((t^{p^{-\infty}}))$, $X_2 = \operatorname{Spa} \mathbb{F}_p((u^{p^{-\infty}}))$, then

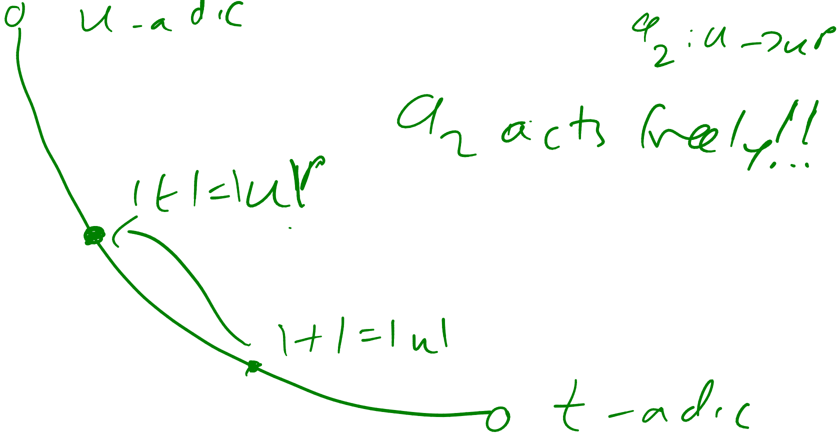
$$X_1 \times X_2 = \{v \in \operatorname{Spa} \mathbb{F}_p[[t,u]][t^{-p^{\infty}}, u^{p^{-\infty}}]^{\wedge}_{(t,u)}[t^{-1}u^{-1}] : v(t), v(u) < 1\},$$

which is *not* quasicompact!



Quotients by partial Frobenius

For $X_1, X_2 \in \mathbf{Pfd}$, put $X = X_1 \times X_2$. This space admits partial Frobenius operators φ_1, φ_2. Unlike for schemes, however, X/φ_2 is an object of $\mathbf{Pfd}$! Moreover, if X_1, X_2 are quasicompact, then so is X/φ_2.

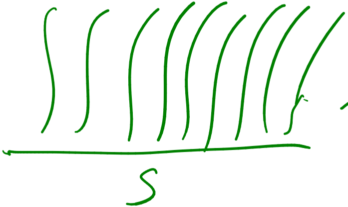


Product with a geometric point

Theorem

For X_2 a geometric point, $\mathbf{FEt}(X_1) \to \mathbf{FEt}(X/\varphi_2)$ is an equivalence.

This reduces to the case where X_1 is itself a geometric point. When $X_2 = \operatorname{Spa} \mathbb{C}_p^\flat$, this can be proved by interpreting X/φ_2 in terms of the Fargues-Fontaine curve for X_1.



$S \times \underline{\mathrm{Spd}(\mathbb{C}_p)}$

$$Y_S \xrightarrow{\varphi} X_S$$

$X_S \times_{\mathbb{C}_p} \mathbb{F}_p$

relative Fargues-Fontaine curve $= X_S$

Product with a geometric point

Theorem

For X_2 a geometric point, $\mathbf{FEt}(X_1) \to \mathbf{FEt}(X/\varphi_2)$ is an equivalence.

For general $X_2 = \operatorname{Spa} K$, we reduce from K to K' where K is a completion of $K'(t)$. A direct calculation rules out abelian covers; one then uses p-adic differential equations to construct a "ramification filtration" to reduce to the abelian case.

Products of two (or more) fundamental groups

Corollary

For $X_1, X_2 \in \mathbf{Pfd}$ connected qcqs, X/φ_2 is connected. For $\overline{x} \to X$ a geometric point,

qcqs

$$\pi_1^{\mathrm{prof}}(X/\varphi_2, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x}).$$

A similar statement holds for diamonds. This can be used to describe p-adic representations of $\pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x})$ in terms of multivariate (φ, Γ)-modules (see Carter–KSK–Zábrádi).

$\mathrm{Rep}^{\mathrm{cnts}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \longleftrightarrow \{(\varphi, \Gamma)\text{-modules}\}$

$[K:\mathbb{Q}_p] < \infty$ $\quad G_K$ $\qquad G_{\mathbb{Q}_p} \times G_{\mathbb{Q}_p}$

Products of two (or more) fundamental groups

Corollary

For $X_1, X_2 \in \mathbf{Pfd}$ *connected qcqs,* X/φ_2 *is connected qcqs. For* $\overline{x} \to X$ *a geometric point,*

$$\pi_1^{\mathrm{prof}}(X/\varphi_2, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x}).$$

When $X_1 = X_2$, are p-adic representations of $\pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x})$ related to vector bundles on the (relative) square of the relative Fargues–Fontaine curve? And how to classify the latter?

More questions

Is there a version for constructible sheaves? (~~See Fargues–Scholze?~~)

Does this build towards an "$\ell = p$" Langlands correspondence for $\mathbb{Q}_p$?

Contents

References for this section

Work in progress, but see...

Abe, Langlands correspondence for isocrystals and the existence of crystalline companions for curves, *J. Amer. Math. Soc.* **31** (2019).

KSK, Notes on isocrystals, arXiv:1606.01321v5 (2018).

KSK, Étale and crystalline companions, I, arXiv:1811.00204v3 (2020).

KSK, Étale and crystalline companions, II, arXiv:2008.13053v1 (2020).

Context: the Langlands correspondence again

Let X be a curve over a finite field of characteristic p. For a given reductive group G, the Langlands correspondence for G is supposed to involve not just $\overline{\mathbb{Q}}_\ell$-sheaves for primes $\ell \neq p$, but also some "crystalline" replacement for $\ell = p$.

This happens in a "de Rham-style" Weil cohomology. The analogue of lisse sheaves are **overconvergent F-isocrystals**. (Today we'll talk about a simpler construction: **convergent F-isocrystals**.)

The analogue of constructive sheaves are **arithmetic $\mathcal{D}$-modules**. Using these, Abe handles the case $G = \mathrm{GL}(n)$ after L. Lafforgue. (I won't define these today.)

We need a form of Drinfeld's lemma to follow the approach of V. Lafforgue for more general G. What we get today won't be enough (because it won't include arithmetic $\mathcal{D}$-modules), but it's progress...

Convergent F-isocrystals

$= \mathrm{Spf}\, W(k)\langle x_1, \dots, x_n \rangle$ $\qquad \mathbb{A}^n_k \rightsquigarrow \widehat{\mathbb{A}}^n_{W(k)}$

Let X be a smooth affine scheme over a perfect field k of characteristic p. Fix a formal scheme P smooth over $W(k)$ with $P_k \cong X$ and a lift σ of φ_X to P.

affine

cf. Kedlaya-Arabia

A **convergent** F-**isocrystal** on X is a finite projective module over $\Gamma(P, \mathcal{O})[p^{-1}]$ equipped with an integrable $W(k)[p^{-1}]$-linear connection and a horizontal isomorphism with its σ-pullback.

The resulting $\mathbb{Q}_p$-linear tensor category $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ does not depend on P or σ, and extends by glueing to general smooth X. We refer to the σ-action also as the φ_X-action.

convergent F-isocrystal $\approx$ lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X

convergent isocrystal $\approx$ $\cdots$ $\cdots$ $X_{\mathbb{F}_p}$

Newton polygons

For $\mathcal{E} \in \mathbf{F}\text{-}\mathbf{Isoc}(X)$ and $\overline{x} \to X$ a geometric point, we may pull back $\mathcal{E}$ to $\mathbf{F}\text{-}\mathbf{Isoc}(\overline{x})$ and apply the Dieudonné–Manin classification: that pullback decomposes as $\bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$ where for $d = \frac{r}{s} \in \mathbb{Q}$ in lowest terms, $\mathcal{E}_d$ admits a basis killed by $\varphi_X^d - p^r$.

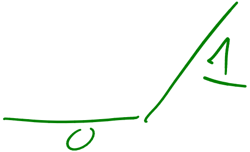
d occurs as a slope with multiplicity rank $\mathcal{E}_d$

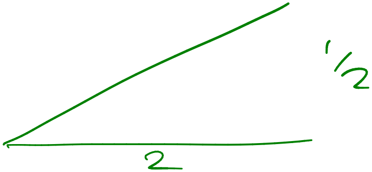
Theorem (Grothendieck–Katz)

The Newton polygon function on $|X|$ is upper semicontinuous.

ordinary

supersingular





Slope filtrations

Theorem (Katz)

If the Newton polygon is constant, then $\mathcal{E}$ admits a filtration

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

in which $\mathcal{E}_i/\mathcal{E}_{i-1}$ has all Newton slopes equal to μ_i, and $\mu_1 < \cdots < \mu_l$.

Theorem (Katz-Crew)

If $\mathcal{E}$ has all Newton slopes equal to 0, then $\mathcal{E}^{\varphi_X}$ is a lisse $\mathbb{Q}_p$-sheaf on X.

Convergent Φ-isocrystals

For $i = 1, 2$, let X_i be a smooth affine scheme over a perfect field k_i of characteristic p. Fix a formal scheme P_i smooth over $W(k_i)$ with $(P_i)_{k_i} \cong X_i$ and a lift σ_i of φ_{X_i} to P_i.

A **convergent Φ-isocrystal** on $X = X_1 \times_{\mathbb{F}_p} X_2$ is a finite projective module over $\Gamma(P_1 \times_{\mathbb{Z}_p} P_2, \mathcal{O})[p^{-1}]$ equipped with an integrable $W(k_1 \otimes_{\mathbb{F}_p} k_2)[p^{-1}]$-linear connection and commuting horizontal isomorphisms with its σ_i-pullbacks. Let $\Phi\,\mathbf{Isoc}(X)$ be the resulting category. (this is abelian!)

$\longrightarrow F\text{-Isoc}(X)$

uses Drinfeld's lemma!

Total Newton polygons

We map $\Phi\,\mathbf{Isoc}(X)$ to $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ by keeping the action of $\varphi = \varphi_1 \circ \varphi_2$.

Theorem

Suppose that X_2 is a geometric point. For $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$, the ***total Newton polygon*** *of $\mathcal{E}$ (i.e., the Newton polygon of the image object in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$) factors through $|X_1|$.*

$|X| \dashrightarrow |X_1|$

Idea of proof: by Grothendieck–Katz, we may apply Drinfeld's lemma to the total Newton polygon stratification.

$|X_1| \times \underbrace{|X_2|}_{\text{point}}$

NP

$\mathbb{Q}^n$

Relative Dieudonné–Manin

Theorem

Suppose that X_2 is a geometric point. Then any $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$ decomposes as $\bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$ where for $d = \frac{r}{s} \in \mathbb{Q}$ in lowest terms, $\mathcal{E}_d^{\varphi_2^{d} - p^r} \in \mathbf{F}\text{-}\mathbf{Isoc}(X_1)$.

s

Idea of proof: first do the case where the total Newton polygon is constant. Then use:

$X_1 \longrightarrow X/\varphi_2 \longleftarrow$ "internal action of Φ_2"

preserves chondery

Theorem

For $U_i \subseteq X_i$ open dense and $U = U_1 \times U_2$, $\Phi\,\mathbf{Isoc}(X) \to \Phi\,\mathbf{Isoc}(U)$ is fully faithful.

(de Jong, KSK)

$F\,\mathrm{Isoc}(X_1) \xrightarrow{\cancel{\cong}} \Phi\,\mathrm{Isoc}(X)$

Products of two (or more) schemes

Theorem (not just a corollary!)

Any irreducible $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$ is a subobject of $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ for some $\mathcal{E}_i \in \mathbf{F}\text{-}\mathbf{Isoc}(X_i)$.

In general, we cannot write $\mathcal{E} = \mathcal{E}_1 \boxtimes \mathcal{E}_2$; think of irreducible representations of product groups.

Again, we first do the case where the total Newton polygon is constant, then use the full faithfulness of restriction.

$\varphi,$

$\mathcal{O}(d_1) \boxtimes \mathcal{O}(d_2)$

$\mathcal{O}(\tfrac{1}{2}) \boxtimes \mathcal{O}(1/2)$

Footnotes

Similar statements (definitely!) apply to **overconvergent F-isocrystals**, and to **logarithmic convergent F-isocrystals**.

One can (probably!) relax the smoothness hypothesis on X_i by some descent arguments. This should even allow X_i to be an algebraic stack (crucial for moduli of shtukas).

One can (hopefully?) also consider constructible objects.

One can (maybe?) give an analogue of the isomorphism

$$\pi_1^{\mathrm{prof}}(X/\varphi_2, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x})$$

in terms of Tannakian fundamental groups.

One can (???) consider isocrystals without Frobenius structure.

This page intentionally left blank