Angle ranks of abelian varieties

Joint work with Taylor Dupuy (Vermont) and David Zureick-Brown (Emory). Preprint available as arXiv:2112.02455v1.

This live whiteboard can be viewed at https://miro.com/app/board/uXjVO9Ngezk=/. Slides will also be posted afterwards at https://kskedlaya.org/slides/ and https://agstanford.com/.

This talk is **hybrid:** I am on the Stanford campus today. Join us for lunch after the talk!

The UC San Diego campus sits on unceded ancestral land of the Kumeyaay Nation. The Kumeyaay people continue to have an important and thriving presence in the region.

Financial support:

UC San Diego Warschawski Professorship

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Abelian varieties over finite fields

A = abelian variety / $\mathbb{F}_q$ of dimension g

Weil polynomial = charpoly $(\mathrm{Frob}_q, H^1(A))$

$$T^{2g} + \underset{\in \mathbb{Z}}{a_1} T^{2g-1} + \cdots \underset{\in \mathbb{Z}}{a_g} T^g + q\underset{\in \mathbb{Z}}{a_{g-1}} T^{g-1} + \cdots + q^g$$

roots over $\mathbb{C}$ $\alpha_1, \ldots \alpha_{2g}$

$$\alpha_{g+i} = \frac{\overline{\alpha_i}}{\alpha_i} \qquad \boxed{|\alpha_i| = q^{1/2}} \quad \text{Weil}$$

Honda–Tate theorem: "basically" a 1-1 correspondence

$$\left\{\begin{matrix}\text{isogeny classes of} \\ AV/\mathbb{F}_q\end{matrix}\right\} \longleftrightarrow \{q\text{-Weil polynomials}\}$$

Abelian varieties in the LMFDB

https://www.lmfdb.org/Variety/Abelian/Fq/

Dupuy
K
Roe
Vincent

LMFDB

⌂ → Abelian varieties → Fq

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Isogeny classes of abelian varieties over finite fields

Introduction

Overview Random
Universe Knowledge

L-functions

Rational All

Modular forms

Classical Maass
Hilbert Bianchi

Varieties

Elliptic curves over $\mathbb{Q}$
Elliptic curves over $\mathbb{Q}(\alpha)$
Genus 2 curves over $\mathbb{Q}$
Higher genus families
Abelian varieties over $\mathbb{F}_q$

Fields

Number fields
p-adic fields

Representations

Dirichlet characters
Artin representations

Groups

Galois groups
Sato-Tate groups

Database

The database currently contains 2,945,722 isogeny classes of abelian varieties of dimension up to 6 over finite fields. You can browse further statistics or create your own.

Browse

By dimension: 1 2 3 4 5 6

By base field cardinality: 2 3 4 5 7 8 9 16 17 19 23 25 27-211 223-1024

Some interesting isogeny classes or a random isogeny class

A table by dimension and base field.

Search

Advanced search options

Cardinality of the Base field	81	e.g. 81 or 3-49	Primitive	
Characteristic of the Base field	3	e.g. 3 or 2-5	Simple	
Dimension	2	e.g. 2 or 3-5	Geometrically simple	
Initial coefficients	[2, -1, 3, 9]	e.g. [2, -1, 3, 9]	Principally polarizable	
p-rank	2	e.g. 2	Jacobian	
Results to display	50			

Display: List of isogeny classes | Counts table | Random isogeny class

Find

Label or polynomial: 2.16.am_cn — Find

e.g. 2.16.am_cn or 1 - x + 2x^2 or x^2 - x + 2

Learn more

Source and acknowledgments
Completeness of the data
Reliability of the data
Labeling convention

This project is supported by grants from the US National Science Foundation, the UK Engineering and Physical Sciences Research Council, and the Simons Foundation.

Angle ranks in the LMFDB

Data by Dupuy, Kedlaya, Roe, Vincent

⌂ → Abelian varieties → Fq → 6 → 2 → al_cj_aio_wq_abtw_cuy

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LMFDB

Abelian variety isogeny class 6.2.al_cj_aio_wq_abtw_cuy over $\mathbb{F}_2$

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Invariants

Base field: $\mathbb{F}_2$

Dimension: 6

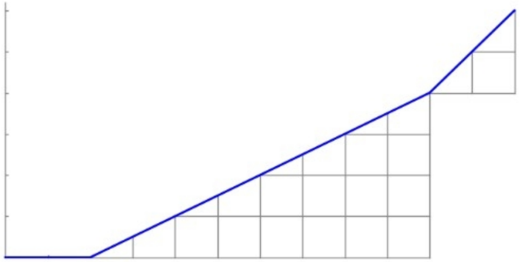
L-polynomial: $(1-2x+2x^2)^4(1-3x+5x^2-6x^3+4x^4)$
$1-11x+61x^2-222x^3+588x^4-1192x^5+1896x^6-2384x^7+2352x^8-1776x^9+976x^{10}-352x^{11}+64x^{12}$

Frobenius angles: $\pm0.123548644961, \pm0.250000000000, \pm0.250000000000, \pm0.250000000000, \pm0.250000000000, \pm0.456881978294$

Angle rank: 1 (numerical)

This isogeny class is not simple, primitive, not ordinary, and not supersingular. It is principally polarizable.

Newton polygon

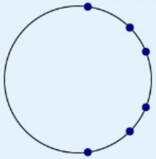


p-rank: 2

Slopes: $[0,0,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1,1]$

Properties

Label 6.2.al_cj_aio_wq_abtw_cuy



Base field	$\mathbb{F}_2$
Dimension	6
p-rank	2
Ordinary	no
Supersingular	no
Simple	no
Geometrically simple	no
Primitive	yes
Principally polarizable	yes
Contains a Jacobian	no

Downloads

All stored data to text
Underlying data

Learn more

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The Newton polygon of an abelian variety

$T^{2g} + a_1 T^{2g-1} + \dots$ $\quad T \xi^s \quad q = p^r$

Newton poly for p-adic valuation, normalized so $v(q) = 1$

Elliptic curves over $\mathbb{Q}$

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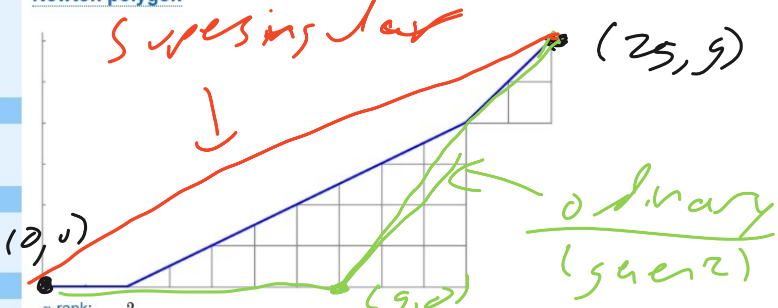
Newton polygon



p-rank: 2

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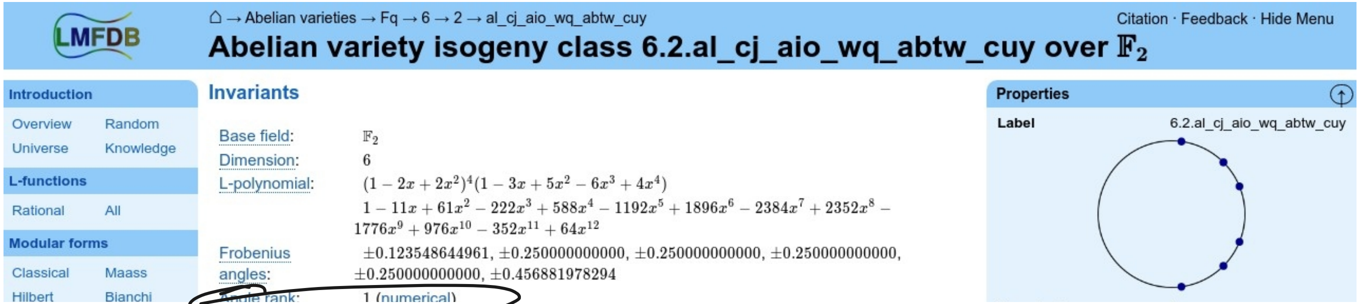
The angle rank of an abelian variety


⌂ → Abelian varieties → Fq → 6 → 2 → al_cj_aio_wq_abtw_cuy

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Abelian variety isogeny class 6.2.al_cj_aio_wq_abtw_cuy over $\mathbb{F}_2$

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L-polynomial:	$(1-2x+2x^2)^4(1-3x+5x^2-6x^3+4x^4)$ $1-11x+61x^2-222x^3+588x^4-1192x^5+1896x^6-2384x^7+2352x^8-1776x^9+976x^{10}-352x^{11}+64x^{12}$
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Angle rank:	1 (numerical)

Properties

Label	6.2.al_cj_aio_wq_abtw_cuy

$\alpha_1 \dots \alpha_{2g} \in \mathbb{C}$ Frobenius eigenvalues

$\alpha_i \alpha_{g+i} = q$

angle rank $:= \mathrm{rank}_{\mathbb{Z}} \left[\dfrac{\alpha_1^{\mathbb{Z}} \alpha_2^{\mathbb{Z}} \cdots \alpha_{2g}^{\mathbb{Z}}}{q^{\mathbb{Z}}} \right]$

$= \mathrm{rank} \left[\dfrac{\mathbb{Z}\arg(\alpha_1) + \cdots + \mathbb{Z}\arg(\alpha_{2g})}{\mathbb{Z}2\pi} \right]$

Angle ranks and the Tate conjecture

Tate conj: the eigenvalue q^i on $H^{2i}(A)$ is entirely explained by cycle classes

True for $i=1$ (Tate)

also true for any A for which all q^{*}-eigenvalues are "generated in codimension 1"

$\Leftrightarrow$ angle rank $= g$ [generic]

(Zarhin)

Example: A supersingular $\Leftrightarrow$ angle rank 0

A theorem of Tankeev

$g = \dim A$

A absolutely irreducible from now on

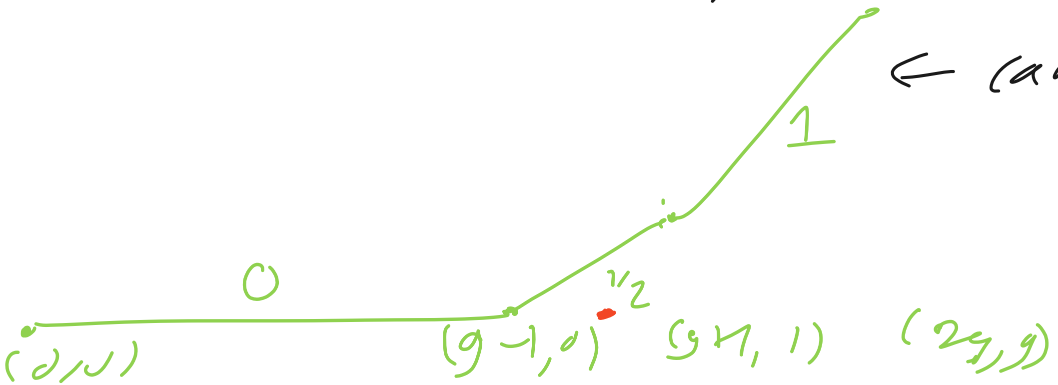
$\underline{\text{Thm}}$ (Tankeev, 1984) If g is $\underline{\text{prime}}$, then

angle rank of $A \in \{1, g-1, g\}$

these can occur

A theorem of Lenstra-Zarhin

A is almost ordinary if its Newton polygon is



$\leftarrow$ codim 1 in moduli

Thm (Lenstra-Zarhin, 1993) If A is almost ordinary, then
$$\begin{cases} g \text{ even: } & \text{angle rank} = g \\ g \text{ odd: } & \text{angle rank} \geq g-1 \end{cases}$$

Comment: also true if NP slopes look like this "2-adically"
$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \boxed{\frac{1}{2}, \frac{1}{2}}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}$

Slope vectors and the angle rank

$V \subset \mathbb{Q}^g$ subspace spanned by

$$\left(v(\beta_1), \ldots, v(\beta_g) \right)$$

for each valuation v of $\mathbb{Q}(\beta_1, \ldots, \beta_g)$ above p.

$$\boxed{\beta_i = \frac{\alpha_i}{\bar{\alpha}_i}}$$

$$\boxed{\dim V = \text{angle rank}}$$

$G = \mathrm{Gal}(\mathbb{Q}(\alpha_1, \ldots, \alpha_g)) \circlearrowright V$

acts by <u>signed</u> permutation matrices.

The Galois action and the code

$G = \mathrm{Gal}(\mathbb{Q}(\alpha, \ldots)/\mathbb{Q})$

$$\begin{tikzcd}
1 \arrow[r] & C \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \overline{G} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}_2^g \arrow[r] & \mathbb{Z}_2^g \rtimes S_g \arrow[r] & S_g \arrow[r] & 1
\end{tikzcd}$$

action on $\beta_1 \ldots \beta_g$

C = code of A. "binary linear code"

G acts on V.

constraints on dimensions of G-reps $\Rightarrow$
constraints on angle ranks given g.
(e.g. Tankeev)

A simple $\Rightarrow$ $\overline{G}$ is transitive

Effects of the code on the angle rank

Suppose $\bar{G}$ acts primitively on $\{1, \ldots, g\}$

if C nontrivial $\quad\quad C \supset (1, 1, \ldots, 1)$

(i.e. $C \neq \langle (1, 1, \ldots, 1) \rangle$) $\quad$ corresponding to

then A has $\quad$ angle rank $= g$ $\quad$ (cplx conj.)

(generically $\quad C = \mathbb{Z}_2^g$

$\bar{G} = S_g$)

Effects of the code on the angle rank

Thm (effective Zarhin)

A = abs simp ab var $/\mathbb{F}_q$ dim g

Frob eigenvalues $\alpha_1 \dots \alpha_{2g}$

$G = \mathrm{Gal}(\mathbb{Q}(\alpha_1 \dots \alpha_{2g})/\mathbb{Q})$

δ = angle rank.

Then vectors $(e_1 \dots e_{2g}) \in \mathbb{Z}^{2g}$ for which $\alpha_1^{e_1} \dots \alpha_{2g}^{e_{2g}} \in \mathbb{Q}^\times$

are generated by vectors of weight at most

$$|G|(|G|-\delta)^3(g\delta)^\delta \qquad |G| \le 2^g g!$$

TBD (if time permits)

Mike:

Hedge for all (M av's are (
⇒ Take for all av's are f.f.

+ improvement...