18.726: Relation between Čech and sheaf cohomology

Hartshorne proves the acyclicity of quasi-coherent sheaves on affine schemes only in the case of a noetherian scheme; Grothendieck's original proof admittedly uses more homological algebra, but its underlying idea is quite nice and explicit, and even somewhat computational (uncharacteristically so for Grothendieck!). So here goes. (Most likely, I will only explain this in lecture for $r = 2$, just to keep indices under control.)

Reference: EGA III.1.1–3.

The exterior algebra complex

Let $X = \operatorname{Spec} A$ be an affine scheme, suppose $f_1, \dots, f_r \in A$ generate the unit ideal, and let $\{U_i = \operatorname{Spec} A_{f_i}\}_{i=1}^r$ be the corresponding open cover of X. Write $f = (f_1, \dots, f_r)$. Define the *exterior algebra complex* $K_\cdot(f)$ associated to f as follows. Take the exterior algebra $\wedge^\cdot(A^r)$ graded in the usual fashion (so $\wedge^0 = A, \wedge^1 = A^r, \dots$). Now view "multiplication by f" as an element of the dual $(A^r)^\vee$; you can then contract with it to get a map $d_f : \wedge^i A^r$ to $\wedge^{i-1} A^r$, and doing it twice gives you zero. So I have a complex, modulo the fact that d goes the wrong way. Never mind for now.

Aside: for $r = 1$, if you turn this around, you just have the complex $0 \to A \stackrel{\times f}{\to} A \to 0$. For general r, you have the "tensor product" of these (i.e., you make an r-dimensional complex, then "flatten" using carefully chosen signs). More explicit description (again, this is turned around so that d goes the right way, so the 0-term above becomes the r-term here and so on): in degree $p \in \{0, \dots, r\}$, you put tuples $(c_{i_0,\dots,i_p})$ of elements of A, one for each $(p+1)$-element sequence $i_0 < \cdots < i_p$ in $\{1, \dots, r\}$. Under d, the tuple maps to the new tuple whose component at the $(p+2)$-element sequence $j_0 < \cdots < j_{p+1}$ is

$$\sum_{l=0}^{p+1} (-1)^l f_l c_{j_0,\dots,\widehat{j_l},\dots,j_{p+1}}.$$

So it looks just like a Čech complex except for that multiplication by f_l.

Given $g = (g_1, \dots, g_r) \in A^r$, I can wedge with g (on the left, say) to get a map $e_g : \wedge^i A^r \to \wedge^{i+1} A^r$. Exercise: this map satisfies

$$(d_f e_g + e_g d_f)m = (f_1 g_1 + \cdots + f_r g_r)m,$$

that being ordinary scalar multiplication on the right. In particular, multiplication by $f_1 g_1 + \cdots + f_r g_r$ is homotopic to zero. But we assumed $f_1, \dots, f_r$ generate the unit ideal, so we can choose $g_1, \dots, g_r$ so that $f_1 g_1 + \cdots + f_r g_r = 0$. Conclusion: the identity map is homotopic to zero, and likewise after applying any functor!

Direct limits

Given an A-module M, write

$$K^\cdot(f, M) = \operatorname{Hom}_A(K^\cdot(f), M);$$

this is a new complex, now with d going in the right direction, and the identity map is again homotopic to zero, so the complex is exact.

Note that for any positive integer n, the tuple $f^n = (f_1^n, \dots, f_r^n)$ also generates the unit ideal, so the identity map in $K^\cdot(f^n)$ is also homotopic to zero. If I now take the direct limit

$$C^\cdot((f), M) = \lim_{\rightarrow} K^\cdot(f^n, M)$$

over all n, I again get an exact complex because cohomology commutes with direct limits. (The homotopy between the identity and zero dies because you can't set it up consistently, but like Pheiddipides, it does so having delivered its intended result, the exactness of $C^\cdot((f), M)$.)

Relation to Čech cohomology

What does this have to do with Čech cohomology?

Put $\mathcal{F} = \tilde{M}$. For $h \in A$, note that $M_h = \Gamma(\operatorname{Spec} A_h, \mathcal{F})$ can be thought of as the direct limit of the system

$$M \stackrel{\times h}{\to} M \stackrel{\times h}{\to} \cdots;$$

that is, if we label the terms $M_h^{(0)}, M_h^{(1)}, \dots$, then roughly $M_h^{(i)}$ corresponds to fractions of the form m/h^i for $m \in M$. (The "roughly" is because two such fractions may not be equal in $M_h^{(i)}$ but may become equal in M_h, because they become equal later in the system.)

The point is that one may canonically identify the Čech p-chains, for the covering $\{U_i\}$, with $C^{p+1}((f), M)$! (That amounts to an exercise in disentangling notation.) In other words, the "extended Čech complex", in which you also stick in the 0-fold intersection sections (namely M) in degree -1, is isomorphic to the complex $C^\cdot((f), M)$ shifted by one.

But I already know that $C^\cdot((f), M)$ is exact, so I am led to the following conclusion.

Theorem 1 (part of EGA III.1.2.4) *Let $X = \operatorname{Spec} A$ be an affine scheme, suppose $f_1, \dots, f_r \in A$ generate the unit ideal, and let $\{U_i = \operatorname{Spec} A_{f_i}\}_{i=1}^r$ be the corresponding open cover of X. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the Čech cohomology groups $\check{H}^i(\{U_i\}, \mathcal{F})$ vanish for $i > 0$.*

Cleanup

That's not yet what I want, but it's close.

Theorem 2 *Let $X = \operatorname{Spec} A$ be an affine scheme. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the sheaf cohomology groups $\check{H}^i(X, \mathcal{F})$ vanish for $i > 0$.*

(Consequence from last lecture: you can compute sheaf cohomology of a quasi-coherent $\mathcal{O}$-module on any separated scheme, by using a Čech complex associated to an open affine covering. The thing you need is that the sheaf cohomology vanishes on any finite intersection of opens in the covering, but separatedness forces that intersection to be again affine.)

How does this follow from my previous theorem? On the homework, you will prove the following result.

Proposition 3 *Let X be a topological space, and let B be a basis of X which is closed under finite intersections (including the empty intersection, so that $X \in B$). Let $\mathcal{F}$ be a sheaf of abelian groups on X, and suppose that $\check{H}^i(U, \mathcal{F}) = 0$ for all $U \in B$ and all $i > 0$, where $\check{H}^i$ denotes the limit of Čech cohomology over all coverings. Then for all $i > 0$, $H^i(X, \mathcal{F}) = 0$.*

That does it, modulo the fact that we didn't consider *all* coverings of an affine scheme, only finite ones by distinguished open affines. But these form a basis, and affine schemes are quasi-compact, so any covering can be refined to a finite covering by distinguished open affines. So indeed $\check{H}^i(U, \mathcal{F}) = 0$ for U any open affine and $\mathcal{F}$ quasicoherent, so we obtain the desired conclusion. Hooray!