Crib Sheet: Properties of Morphisms of Schemes

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Surgeon General's Warning: Not reading EGA before using this crib sheet may be hazardous to your health. In particular, certain definitions (e.g., projective) do not match Hartshorne.

Here are some properties of a morphism $f : X \to Y$ of schemes. Abbreviations: LS = (Zariski) local on the source, LT = (Zariski) local on the target/base.

quasicompact: For every affine open $V \subseteq Y$, $f^{-1}(V)$ is a quasi-compact topological space. Equivalent: for every affine open $V \subseteq Y$, $f^{-1}(V)$ is a *finite* union of affine opens of X. [LT]

quasiseparated: The diagonal $\Delta_f : X \to X \times_Y X$ is quasicompact.

separated: The diagonal $\Delta_f : X \to X \times_Y X$ is a closed immersion.

affine: For every affine open $V \subseteq Y$, $f^{-1}(V)$ affine. Equivalent: $X \cong \mathrm{Spec}_Y(\mathcal{A})$ for some quasicoherent $\mathcal{O}_Y$-algebra $\mathcal{A}$. [LT, $\Longrightarrow$ separated]

quasiaffine: f is quasicompact and admits a factorization $\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & X' \\ {}_{f}\searrow & & \swarrow_{f'} \\ & Y & \end{array}$ with j an open immersion and f' affine. Equivalent: for every affine open $V \subseteq Y$, $f^{-1}(V)$ is isomorphic to a quasicompact open subscheme of an affine scheme.

locally of finite type/presentation: For every affine open $V \subseteq Y$, $f^{-1}(V)$ can be covered by affine opens U such that $\Gamma(X, U)$ is a finitely generated/presented $\Gamma(Y, V)$-algebra. [LS]

of finite type/presentation: f is quasicompact and locally of finite type/presentation. [LT]

quasiprojective: f is of finite type and there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that $\mathcal{L}$ is f-ample. Consequence: for every affine open $V \subseteq Y$, a suitably high power of $\mathcal{L}$ has sections over $f^{-1}(V)$ which define an immersion of $f^{-1}(V)$ into $\mathbb{P}^N_V$.

proper: f is separated, of finite type and universally closed. [LT]

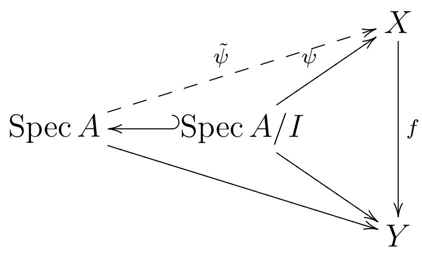
projective: X is isomorphic to $\mathrm{Proj}_Y \mathcal{S}$, where $\mathcal{S}$ is a quasicoherent graded $\mathcal{O}_Y$-module generated in degree 1 such that $\mathcal{S}_1$ is a finitely generated $\mathcal{O}_Y$-module.

quasifinite: f is of finite type, and for all $y \in Y$, the scheme $X_y = f^{-1}(Y)$ is finite over $\mathrm{Spec}\,\kappa(y)$.

finite: f is affine and for every affine open $V \subseteq Y$, the map $\Gamma(Y, V) \to \Gamma(X, f^{-1}(V))$ is finite. Equivalent: f is proper and quasi-finite. [LT]

formally unramified	For all diagrams (1), there exists at most one $\tilde{\psi}$.
formally smooth	For all diagrams (1), there exists at least one $\tilde{\psi}$.
formally étale	For all diagrams (1), there exists exactly one $\tilde{\psi}$.
unramified	f is formally unramified and locally of finite presentation.
smooth	f is formally smooth and locally of finite presentation.
étale	f is formally étale and locally of finite presentation.

In the following diagram, I is an ideal of the ring A such that $I^2 = 0$.



(1)

Some trivial and nontrivial results:

1. A scheme X is (separated/affine) iff $X \to \operatorname{Spec} \mathbb{Z}$ is.
2. f is formally unramified iff $\Omega^1_{X/Y} = 0$.
3. f is étale iff f is flat and unramified.
4. f is smooth iff f is locally of finite presentation, flat and has smooth fibres.
5. If f is of finite type and formally unramified, then f is quasifinite.
6. For any scheme X, the diagonal $\Delta : X \to X \times_{\operatorname{Spec} \mathbb{Z}} X$ is separated and locally of finite type. If X is quasiseparated, then Δ is of finite type and also quasiaffine.
7. If f is quasifinite and separated, then f is quasiaffine.
8. (Zariski's Main Theorem) If f is quasifinite and separated and Y is quasicompact and quasiseparated, then there exists a factorization $f = f' \circ u$ with $u : X \to X'$ an open immersion and $f' : X' \to Y$ finite.
9. (Nagata's Theorem) If f is of finite type and separated and Y is Noetherian, then there exists a factorization $f = f' \circ u$ with $u : X \to X'$ an open immersion and $f' : X' \to Y$ proper. (This presumably also holds with Y quasicompact and quasiseparated.)